Gas of shells as microscopic origin of black holes entropy

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Outlook

- Introduction
  - Four Laws of Thermodynamics vs Four Laws of Black Hole mechanics
  - Third Law of Thermodynamics and its violation by BHs
- Microscopic origin of the Bekenstein-Hawking entropy via Bose gas models
  - entropy of **non-local** Bose gas models with the zeta function regularizations
- BH entropy via random thin shell model
  - entropy of random thin shell models

Four Laws of Thermodynamics vs Four Laws of Black Hole mechanics

- There is a remarkable analogy between the laws of thermodynamics and the laws of black hole mechanics

Thermodynamics

- 0. $E, T, S, V, P, ...$
- 1. $dE = TdS - PdV$
- 2. $\delta S \geq 0$
- 3. $S \to 0$ if $T \to 0$

Black Hole mechanics
(Bardeen, Carter, Hawking,73'; Bekenstein 73')

- 0. surface gravity $\kappa = \frac{1}{M}$, $Q, a, ...$
- 1. $dM = \frac{1}{8\pi M} d\,\frac{\mathcal{A}}{4} + ...$
- 2. $\delta \mathcal{A} \geq 0$
- 3. States with $\kappa = 0$ are unattainable

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- A missing link in this area is a precise statistical mechanical interpretation of entropies for all varieties of black holes.
- We can try to find a statistical mechanics model with the same dependence of entropy on other thermodynamic variables as a particular black hole has
- However, there is a problem with the third law of thermodynamics

Third Law of Thermodynamics

- **In the Planck formulation** : Entropy $S \to 0$ as $T \to 0$ ($\beta = \frac{1}{T} \to \infty$)
- In the Nernst formulation

$$\delta S(T,x) \equiv S(T,x) - S(T,x') \to 0 \quad as \quad T \to 0 \tag{1}$$

or

$$\lim_{T\to 0} S(T,x) - \text{universal constant}$$

- Unattainability of $T = 0$

REFS: W.Israel, 1986; R.Wald, 1997;
F. Belgiorno and M. Martellini, 2004;
C. Kehle and R. Unger, 2211.1574.

Violation of Third Law in BH Thermodynamics

- **Schwarzschild black hole**
  - Hawking temperature $T = \frac{1}{8\pi M}$
  - Bekenstein-Hawking entropy $S = \frac{1}{16\pi T^2} \to \infty \quad as \quad T \to 0$

    Violation in Planck formulation

- Reissner-Nordstrom black hole
  - Hawking temperature $T = \frac{\sqrt{M^2 - Q^2}}{2\pi\left(\sqrt{M^2-Q^2}+M\right)^2} \to 0$ for $M \to Q$ or $M \to \infty$
  - BH entropy $S = \pi\left(\sqrt{M^2 - Q^2} + M\right)^2 \to \pi Q^2$ for $T \to 0$ **depends on** Q
- Kerr

    Violation in Nernst formulation

Physical systems with violation of the Third Law *

- Lattice models.
  The question of whether the third law is satisfied can be decided completely in terms of ground-state degeneracies
  M. Aizenman, El. Lieb 80'

- Ice models.
  V. F. Petrenko and R. W. Whitworth, 99', *Physics of Ice*

- Strange metals.
  J. Zaanen et al. 15', *Holographic duality in condensed matter physics.*

Few Refs. on microscopic origin of BH entropy *

- **The problem of the microscopic origin of the Bekenstein-Hawking entropy of a black hole has attracted a lot of attention over the past 30 years**

  - **Wheeler** considered of the BH interior as "bag of gold" (**Almheiri et al 20**)

  - **Strominger and Vafa, 96'** $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$, $f(r) = \left(1 - (\frac{r_0}{r})^2\right)^2$, $r_0 = (\frac{8Q_H Q_F^2}{\pi^2})^{1/6}$, $S_{BH} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}$
  D-0 branes interpretation: $d(n,c) \sim \exp(2\pi\sqrt{\frac{nc}{6}})$, $c = 6(\frac{1}{2}Q_F^2 + 1)$, $\quad n = Q_H$

$$S_{stat} = \ln d(Q_F, Q_H) \sim 2\pi\sqrt{Q_H(\frac{1}{2}Q_F^2 + 1)}$$

  - **'t Hooft 84'** proposed to relate BH entropy with the entropy of thermally excited quantum fields in the vicinity of the horizon.

  - Recent searches **Balasubramanian et al 22'** for internal geometries that provide the entropy of BH.

  - Matrix models corresponding to BH in spacetime with topology $AdS_2 \times S^8$, **Maldacena'23**

Summary of Introduction

- **Schwarzschild BHs violate 3-d law of thermodynamics.**

  Schwarzschild BH entropies in D-dim $S \to \infty$ rather than zero when $T \to 0$.

- **We search for quantum statistical models with such exotic thermodynamic behaviour.**

- **A special interest present the models that are related with gravity, i.e. models that contain G_N. We will discuss a special class of such models — thin shells in GR.**

Free energy of non-local Bose gas (NLBG).

- d-dim Bose gas

$$F_{BG}(d,\varepsilon) \quad = \quad \frac{\Omega_{d-1}}{\beta}\int_0^\infty \ln\left(1-e^{-\lambda\,\beta\,\varepsilon(k)}\right)\,k^{d-1}dk$$

  - standard (local case) $\varepsilon(k)=k^2$
  - d-dim **α-non-local** Bose gas $\varepsilon(\mathbf{k})=\mathbf{k}^\alpha$, $F_{BG}(d,\alpha)=F_{BG}(d,\varepsilon)\Big|_{\varepsilon=k^\alpha}$
  - d-dim **$\mathcal{F}$-non-local Bose gas**, $\mathcal{F}(k)$ -an analytical function.
    $\mathcal{F}(k)=\exp(ck^2)$. **SFT: IA, astro-ph/0410443;**
    p-adics: V.S.Vladimirov, see B.Dragovich's talk.

- Explicit form

$$F_{BG}(d,\alpha)=-\frac{2\pi^{d/2}}{d\Gamma(d/2)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right).$$

- Free energy of D-dim Schwarzschild BH $F_{BH}(D,\beta)$ [**see next slides**]

- Our stategy: $F_{BH}(D,\beta)=F_{BG}(d,\beta)$

- Schwarzschild solution

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2,$$

- Hawking temperature and Bekenstein-Hawking entropy

$$T = \frac{1}{8\pi M}, \qquad S = 4\pi M^2 = \frac{\beta^2}{16\pi}$$

- Free energy

$$F = \frac{\beta}{16\pi}$$

D=4 Schwarzschild BH vs Bose Gas. 2/3

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$

$$-\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\,\zeta\left(\frac{d}{2}+1\right) = \frac{\beta}{16\pi} \qquad (*)$$

- To fulfill (*) we have to assume

$$d = -4, \qquad \lambda^2 = -\frac{\pi}{16\,\zeta(-1)}.$$

- Taking into account that $\zeta(-1) = -1/12$, we get

$$\lambda = \sqrt{\frac{3\pi}{4}},$$

D=4 Schwarzschild BH vs Bose Gas. 3/3

- Therefore, we obtain that the thermodynamics of the 4-dim Schwarzschild BH is equivalent to the thermodynamics of the Bose gas in $d = -4$ spatial dimensions.
- We understand the thermodynamics of the Bose gas in negative spatial dimensions in the sense of the analytical continuation of the right hand site of

$$F_{BG} \quad = \quad -\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\,\zeta\left(\frac{d}{2}+1\right).$$

- D-dimensional Schwarzschild black hole, $D \geq 4$,

$$ds^2 = -\left(1 - \frac{{r_h}^{D-3}}{r^{D-3}}\right) dt^2 + \frac{dr^2}{1 - \frac{{r_h}^{D-3}}{r^{D-3}}} + r^2 d\omega_{D-2}^2,$$

- Hawking temperature $T = 1/\beta = \frac{D-3}{4\pi r_h}$
  r_h is the radius of the horizon.
- The entropy and the free energy are

$$S = \frac{\Omega_{D-2}}{4}\left(\frac{D-3}{4\pi}\,\frac{1}{T}\right)^{D-2}; \; F = \frac{(D-3)^{D-3}\beta^{D-3}\,\Omega_{D-2}}{4(4\pi)^{D-2}}$$

$S \to \infty$, when $T \to 0$ — a violation of the 3-d law

- Equalizing: $F_{BG}(\beta) = F_{BH}(\beta)$ series of solutions

I.Volovich's talk

D>4 Schwarzschild BH vs Bose Gas. 2/4

- 4 series of solutions

D	d	α
$D = 4k+1, \quad k = 1, 2, 3...$	$d = (4k-1)\lvert\alpha\rvert$	$\alpha = -q,\ q = 1, 2, 3$
$D = 4k+1, \quad k = 1, 2, 3...$	$d = -(4k-1)\alpha$	$\frac{4r}{4k-1} < \alpha < \frac{2(2r+1)}{4k-1}, \quad r = 0, 1, 2, ...$
$D = 4k+3, \quad k = 1, 2, 3...$	$d = -(4k+1)\alpha$	$\frac{2(2r+1)}{4k+1} < \alpha < \frac{4(r+1)}{4k+1}, \quad r = 0, 1, 2...$
$D = 2k, \quad k = 2, 3, 4...$	$d = -2(k-1)\alpha$	$\alpha = \frac{p}{k-1}, \quad p = 1, 2, ..$

- Euclid $d = 3$
  Kaluza-Klein $d = 5$
  Superstrings $d = 10$
  Here $d < 0$

$$-\left(\frac{L}{2\pi}\right)^{d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda_\alpha}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right)=\frac{(D-3)^{D-3}}{4G_D(4\pi)^{D-2}}\beta^{D-3}\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$$

To equalize the powers of β we take $d=-(D-2)\,\alpha$ and we get

$$F_{BG}=-\left(\frac{2}{L}\right)^{(D-2)\,\alpha}\frac{\pi^{\frac{(D-2)\alpha}{2}}}{\Gamma(1-\frac{(D-2)\alpha}{2})}\beta^{D-1}\lambda_\alpha^{D-2}\underbrace{\Gamma\,(3-D)\,\zeta\,(3-D)}_{\frac{\zeta(D-2)}{2^{D-2}\pi^{D-3}\sin(\frac{\pi(D-2)}{2})}}$$

$$\lambda_\alpha=\Big(\mathcal{B}(D,\alpha)\mathcal{A}(D,\alpha)\Big)^{1/(D-2)}\quad\text{where}\quad \mathcal{B}(D,\alpha)=-\Gamma\left(1-\frac{(D-2)\alpha}{2}\right)\sin\left(\frac{\pi(D-2)}{2}\right)$$

$$\mathcal{A}(D,\alpha)=\frac{L^{\alpha(D-2)}}{G_D}\underbrace{\frac{(D-3)^{D-3}}{\zeta(D-2)\Gamma(\frac{D-1}{2})}}_{>0,forD>4}2^{\cdots}\pi^{\cdots}$$

Since D is a natural number,

$$\sin(\frac{\pi(D-2)}{2})=\begin{cases}1 & \text{for} \quad D=4k+3, \quad k=1,2,3,\ldots\\ 0 & \text{for} \quad D=2k, \quad k=2,3,4,\ldots\\ -1 & \text{for} \quad D=4k+1, \quad k=1,2,3,\ldots\end{cases}.$$

- Let consider the 3rd case. $\Gamma(...)>0\Rightarrow\alpha<0$

- This 3-rd solution corresponds to 1-st line in the table. $D = 4k + 1$. $k = 1, D = 5$ we have solutions with $\alpha = -1, -2, ...$

  $d = -(D-2)\,\alpha.$

  In these cases

$$
\begin{aligned}
d = 3, \quad \alpha = -1, \quad \lambda_{-1} &= \frac{1}{2\sqrt[3]{G_5}L}\sqrt[3]{-\frac{3\pi}{\zeta'(-2)}} = \frac{3.38}{\sqrt[3]{G_5}L}, \\
d = 6, \quad \alpha = -2, \quad \lambda_{-2} &= \frac{2}{\sqrt[3]{G_5}L^2}\sqrt[3]{-\frac{3\pi^2}{\zeta'(-2)}} = \frac{19.814}{\sqrt[3]{G_5}L^2}.
\end{aligned}
$$

Gas of random quantum thin shells

- **A spherical symmetric thin shell $\Sigma = \mathbb{R} \times \mathbb{S}^2$ in spherical symmetric background divides the spacetime on**
  - the internal spacetime, $\mathcal{M}^-$ (with Schw. coord. $(t_-, r_-, \phi\theta)$), and
  - an external spacetime, $\mathcal{M}^+$(with Schw. coord. t_+, r_+, ϕ, θ)
- The shell can be describe by equations

$$r_\pm = r = R(\tau), \quad t_\pm = t(\tau).$$

- In term of intrinsic coord. of the shell (τ, θ, ϕ), the induced metric on Σ is

$$ds^2_\Sigma = d\tau^2 - R^2(\tau)d\Omega^2$$

Berezin, Kusmin,Tkachev,1988

Effective action for the shell

- **The effective action for the shell in the proper time**

$$S=\int d\tau\left[-m+f(R)\sqrt{1+R_\tau^2}\right], \qquad f(R)=\frac{Gm^2}{2R}$$

- **Hamiltonian**

$$H=m-\sqrt{f^2-P^2},$$

- **Wheeler-DeWitt equation**

$$\left[(-i\partial_\tau+m)^2-\partial_R^2-f^2\right]\Psi(\tau,R)=0.$$

Stationary solutions of WdW eq. Spectrum

Taking $\Psi(\tau, R)$ in the form

$$\Psi(\tau, R) = e^{-i\mathcal{E}\tau}\psi(R),$$

we get the stationary version WdW eq.

$$\psi''(R) - \left[(m-\mathcal{E})^2 - \frac{m^4}{4m_p^4 R^2}\right]\psi(R) = 0, \quad (*)$$

m is the shell mass, m_p is the Planck mass, $m_p = 1/\sqrt{G}$, G is the Newton gravitational constant. $\hbar = c = 1$.

Spectrum

The spectrum of equation (*) for $m > m_p$ is **Vaz, 2022**

$$\mathcal{E}_n(m) = m\left(1 - e^{-n\pi/\mathfrak{b}}\right), \qquad (**)$$
$$\mathfrak{b} = \frac{1}{2m_p^2}\sqrt{m^4 - m_p^4}, \quad m_p < m$$

n is a positive integer.

Free energy of bose gas of shells

- The free energy of bose gas of shells at temperature $T = 1/\beta$ and chemical potential μ

$$F_{gas-of-shells}(\beta, \mu, m) = \frac{1}{\beta} \sum_n \ln \left(1 - e^{\beta\,(\mu - \mathcal{E}_n(m))} \right)$$

here $\mathcal{E}_n(m)$, $n = 1, 2, 3, ...$ is the spectrum (**)

Free energy of RANDOM bose gas of shells, 1/4

At temperature $T = 1/\beta$

$$\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p)$$

$$= \frac{1}{\beta} \int \sum_{n}^{N} \ln \left(1 - e^{\beta\,(\mu - \mathcal{E}_n(m))}\right) d\sigma(m)$$

$\mathcal{E}_n(m)$, $n = 1, 2, 3, ...$ is spectrum (**) for fixed **random parameter** m
.

$\mathrm{d}\sigma = d\sigma(m)$ **is the probability measure**

Free energy of random bose gas of shells, 2/4

Now we specify the measure $\sigma = \sigma(m)$ and deal with $\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p)$ given by

$$\mathcal{F}_{gas-of-shells}(\beta, \mu, m_p) = \frac{1}{\beta} \int_{m_p(1+\Delta)}^{2m_p} \sum_{n}^{N} \ln\left(1 - e^{\beta\,(\mu - \mathcal{E}_n(m))}\right) \frac{C\,dm}{(m - m_p)^3},$$

where $\Delta > 0$ is the regularization parameter and the constant C is derive from normalization

$$C^{-1} = \int_{m_p(1+\Delta)}^{2m_p} \frac{dm}{(m - m_p)^3}$$

and for small regularization parameter Δ

$$C = \frac{2\Delta^2 m_p^2}{1 - \Delta^2} \approx 2m_p^2 \Delta^2$$

Free energy of random bose gas of shells, 3/4

After the change of the variable $m - m_p = x m_p$ and taking $\mathcal{E}_n(m) \approx m = m_p(1+x)$ we get the representation

$$\mathcal{F}_{gas.shells}(\beta, \mu, m_p) = \frac{2N\Delta^2}{\beta} \int_\Delta^1 \ln\left(1 - e^{\beta\,(\mu - m_p(1+x))}\right) \frac{dx}{x^3},$$

Taking $\mu = m_p$ we finally get

$$\mathcal{F}_{gas.shells}(\beta, m_p) \approx \frac{2N\Delta^2}{\beta} \int_\Delta^1 \ln\left(1 - e^{-\beta\, m_p x}\right) \frac{dx}{x^3},$$

$$I(a) = \int_a^\infty \ln\left(1 - e^{-x}\right) \frac{dx}{x^3}$$

Free energy of random bose gas of shells, 4/4

Denote $N\Delta^2 = \lambda$ and consider $\Delta \to 0$ and $N \to \infty$,

$$N\beta C\, I(a) = N\beta 2m_p^2\, \Delta^2\, I(a) = 2m_p^2 \lambda\beta\, I(a)$$

Taking the renormalized value of I we get at $a \to 0$

$$\mathcal{F}_{ren,gas-of-shells}(\beta, m_p) \quad \approx \quad 2\,\lambda\, I_{ren}\, m_p^2 \beta$$

and the entropy is equal to

$$S = 2\,\lambda I_{ren}\, m_p^2 \beta^2$$

We set $2\,\lambda I_{ren} = \frac{1}{16\pi}$. This gives the BH entropy

$$S = \frac{1}{16\pi}\, m_p^2 \beta^2 = \frac{1}{16\pi G}\, \beta^2 = 4\pi G M^2$$

Equivalence of ζ-function analytical renormalization and minimal subtraction scheme

- "Cut-off"regularization $I(a)=\int_a^\infty \frac{\log\left(1-e^{-x}\right)}{x^3}\,dx$.

$$I(a)=\frac{\log(a)}{2a^2}+\frac{1}{4a^2}-\frac{1}{2a}-\frac{\log(a)}{24}+\mathbf{0.121}+\mathcal{O}(a), \qquad I_{ren}\Big|_{s=-1}=0.121$$

- ζ-function regularization

$$J(s)=\int_0^\infty \ln(1-e^{-x})\,\frac{dx}{x^{2-s}}=-\Gamma(-1+s)\zeta(s) \tag{2}$$

well defined for $\Re s>1$ and is singular at $s=-1$

$$J(s) = \frac{1}{24(s+1)}+\frac{24\log(A)+1-2\gamma}{48}+\mathcal{O}\left(s+1\right)$$

$$=\frac{0.0416}{s+1} + \mathbf{0.121}+\ldots \quad A - \text{Glaisher const} = 1.282; \quad \gamma - \text{Euler const}$$

$$J_{ren}\Big|_{s=-1} = 0.121, \qquad I_{ren}\Big|_{s=-1}=J_{ren}\Big|_{s=-1}$$

I.A. and I.Volovich, 2305.19827

Conclusion

- Black holes violet the third law of thermodynamics
- Model of bose gas violeting the third law of thermodynamics is proposed
- Random quantum gas of thin shells reproducing the black hole entropy is proposed

- **Analytical regularization**. The starting point

$$I(s)=\int_0^\infty \ln\left(1-e^{-x}\right)\frac{dx}{x^{1+s}}=-\Gamma(-s)\,\zeta(-s+1),\quad \Re s<0 \tag{3}$$

However, the right-hand side of (3) is well defined for all $s\neq 0$ and $s\neq n$, here $n\in\mathbb{Z}_+$ and we denote it by $\mathcal{I}(s)$,

$$\mathcal{I}(s)=-\Gamma(-s)\,\zeta(-s+1). \tag{4}$$

The function $\mathcal{I}(s)$ given by (4) is a meromorphic function for $s\in\mathbb{C}$. It has poles at $s=n>0$ and a double pole at $n=0$. **We define $\mathcal{I}_{ren}(n)$ as**

$$\begin{aligned}\mathcal{I}_{ren}(n) &\equiv \lim_{\epsilon\to 0}\left[-\Gamma(-n+\epsilon)\zeta(1-n+\epsilon)-\text{Pole Part}\left[(-\Gamma(-n+\epsilon)\zeta(1-n+\epsilon)\right]\right]\\ &\quad\ \text{at point}\quad n=1,2,3,...\\ \mathcal{I}_{ren}(0) &\equiv \lim_{\epsilon\to 0}\left[-\Gamma(\epsilon)\zeta(1+\epsilon)-\text{Double Pole Part}\left[(-\Gamma(\epsilon)\zeta(1+\epsilon)\right]\right]\\ \\ \mathcal{I}_{ren}(s) &\equiv \mathcal{I},\quad s>0, s\neq\mathbb{Z}_+\,.\end{aligned}$$

Backup. Equivalence of renormalization

- **Lemma 1.** *The renormalized version of* (3) *after analytical renormalizations is given by*

$$\mathcal{I}_{ren}(n) = -\begin{cases} \frac{(-1)^n}{n!}\left[\zeta'(1-n) + \left(-\gamma + \sum_{k=1}^{n}\frac{1}{k}\right)\zeta(1-n)\right], & n = 1,2,3... \\ \frac{1}{12}\left(12\gamma_1 + 6\gamma^2 - \pi^2\right), & n = 0 \end{cases}$$

Backup. Equivalence of renormalization

- **Cut-off regularizations**. The starting point

$$I(s,a) \equiv \int_a^\infty \ln\left(1-e^{-x}\right)\frac{dx}{x^{1+s}},\ a>0.$$

We find a singular part of the asymptotics of the integral $I(s,a)$ as $a \to 0$ in the form

$$S(s,a) = \sum_{i\geq 0} A_i \frac{\log a}{a^i} + \sum_{i\geq 1} C_i \frac{1}{a^i}.$$

Then we subtract this singular part $S(s,a)$

$$I_{ren}(s,a) = I(s,a) - S(s,a),$$

and finally remove the regularisation

$$I_{ren}(s) = \lim_{a\to 0} I_{ren}(s,a).$$

Backup. Equivalence of renormalization

- **Lemma 2** *The renormalized version of $I(s,a)$ after minimal renormalizations is*

$$\begin{aligned} I_{ren}(s) &= \int_0^1 \frac{1}{x^{s+1}}\left[\ln\left(\frac{1-e^{-x}}{x}\right) - \sum_{k=1}^{n(s)} c_k x^k\right] dx \\ &- \frac{1}{s^2} + \sum_{k=1}^{n(s)} \frac{c_k}{k-s} + \int_1^\infty \frac{1}{x^{s+1}} \ln\left(1-e^{-x}\right) dx\,, \\ n(s) &= \text{Entier}[s], \text{ i.e the integer part of } s. \end{aligned}$$

Backup. Equivalence of renormalization

- **Theorem**. *The minimal renormalized free energyfor $s = n \neq 0$ and the analytic renormalized free energy coincide*

$$I_{ren}(n) = \mathcal{I}_{ren}(n).$$

and

$$I_{ren}(s) = \mathcal{I}(s), \quad \text{for} \quad s > 0 \quad \text{and} \quad s \neq n \in \mathbb{Z}_+.$$