Noether Symmetries in Non-Local Gravity Cosmology

Salvatore Capozziello

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INFN Istituto Nazionale di Fisica Nucleare

SSM Scuola Superiore Meridionale

Università degli Studi di Napoli Federico II

Outline

- *Locality* and *Non-Locality* in Physics
- Non-Local Theories of Gravity
- The Noether Symmetry Approach
- Non-Local Gravity Cosmology
- Astrophysical tests by Galactic Center
- Non-Local Gravity and clusters of galaxies
- Gravitational Waves in Non-Local Gravity
- Conclusions and perspectives

Locality and Non-locality

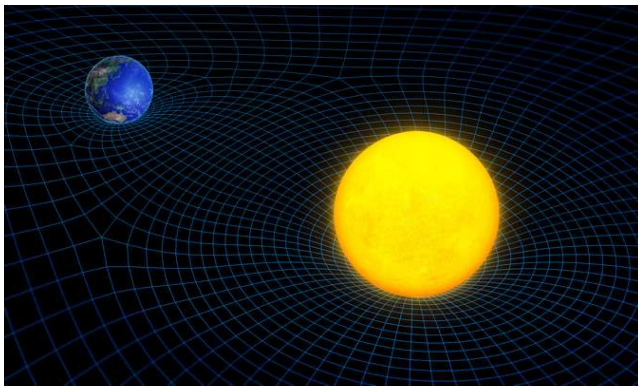
Kinematics

It refers to the *STATES*

Classical Theories: local

CM ⟶ *Points of a tangent/cotangent bundle*

CFT ⟶ *Tensor fields over a manifold*



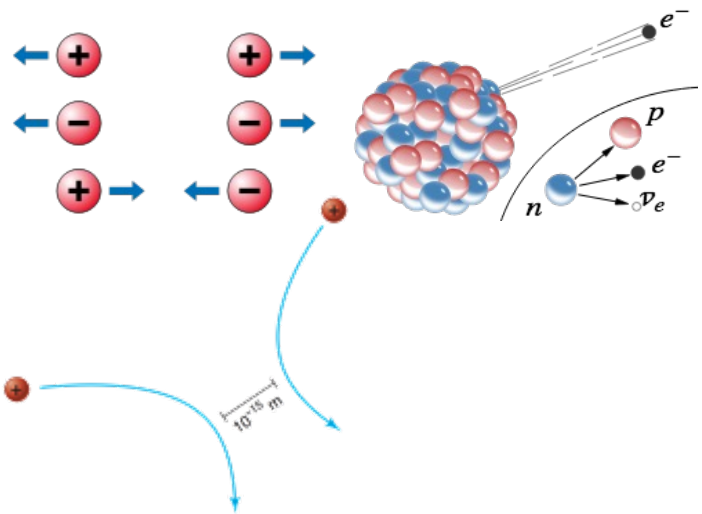
VS

Dynamics

It refers to the *INTERACTIONS*

Quantum Theories: non-local

- *Born interpretation of* Ψ
- *Heisenberg uncertainty principle*



Local Action vs Non-local Action

Local Action

it is a functional of only local fields, *i.e.* algebraic functions of fields or their derivatives evaluated at a single point

It is the paradigm of all fundamental field theories, both classical and quantum

Non-local Action

it is a functional of non-local fields (at least one), *i.e.* functions of fields evaluated at more than one point or transcendental functions of fields or their derivatives

It describes an effective theory

We need a link between GR and QM

GR has to be improved

QFT has to be improved

Nonlocality in physics

Kinematical non-locality in Quantum Mechanics →

- Uncertainty principle
  - No possibility to localize the system during its evolution
  - No unique path despite giving the initial conditions
- Quantum entanglement

Dynamical non-locality in Quantum Field Theory →

- Lagrangians made of non-polynomial differential operators
- Manifests in all fundamental interactions when one-loop effective actions are taken into account
- May arise when QFT on curved spacetime is considered and non-perturbative techniques are used for dimensional regularization

A. O. Barvinsky, "Aspects of nonlocality in quantum field theory, quantum gravity and cosmology," *Modern Physics Letters A*, vol. 30, no. 03n04, p. 1 540 003, 2015, ISSN: 1793-6632. DOI: 10.1142/s0217732315400039.

Non-Locality in Physics

- **Fundamental interactions are non-local. It can be shown by considering the one-loop effective action**

Euler-Heisenberg Lagrangian

$$\mathcal{L}_{EH} = -\frac{1}{4}\mathcal{F}^2 - \frac{e^2}{32\pi^2}\int_0^\infty \frac{ds}{s}\, e^{i\varepsilon s} e^{-m^2 s}\left[\frac{\mathrm{Re}\cosh(esX)}{\mathrm{Im}\cosh(esX)} F_{\mu\nu}F^{\mu\nu} - \frac{4}{e^2 s^2} - \frac{2}{3}\mathcal{F}^2\right]$$

$$\mathcal{F} = \tfrac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right),\; X = \mathcal{F} + i\mathbf{E}\cdot\mathbf{B}$$

Non-Locality in Physics

Yukawa Lagrangian

$$\mathcal{L}_Y = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}\phi(\Box + m^2)\phi + \lambda\,\phi\bar{\psi}\psi$$

The related effective action is

$$\mathcal{L}_{eff} = i\bar{\psi}\partial\!\!\!/\psi + \frac{\lambda^2}{2}\bar{\psi}\psi(\Box + m^2)^{-1}\bar{\psi}\psi$$

The non-locality is in the operator

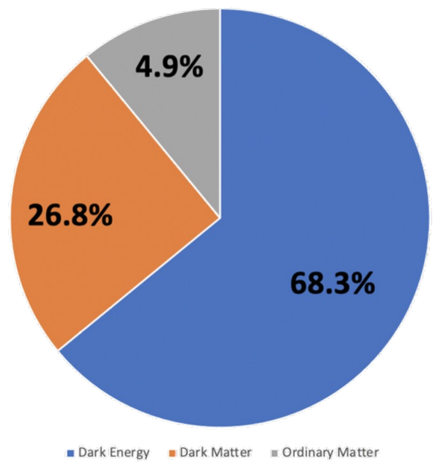
$$(\Box + m^2)^{-1}$$

Non-Locality could fix some General Relativity shortcomings

Large Scales

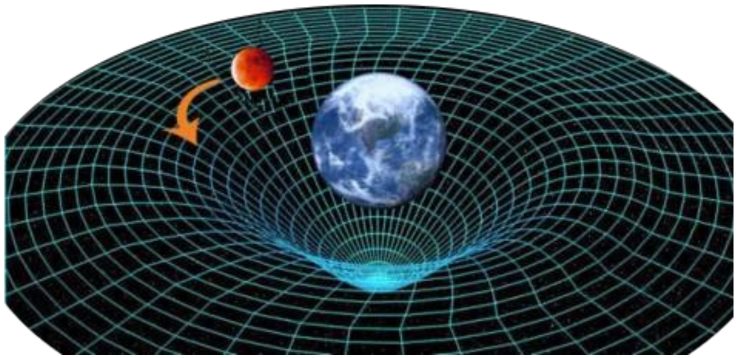
No theory is capable of solving these problems at once so far

- Universe accelerated expansion
- Dark energy
- Galaxy Rotation Curve
- Dark side
- Fine-tuning of cosmological parameters
- Ho tension et al.



Small Scales

- Renormalizability
- GR cannot be quantized
- GR cannot be treated under the same standard as other interactions
- Discrepancy between theoretical and experimental value of Λ
- Spacetime singularities

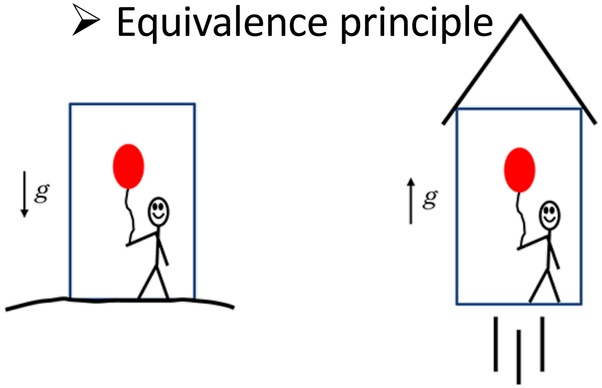


Can Fundamental (UV) and Dark Side (IR) Issues be solved by Non-locality?

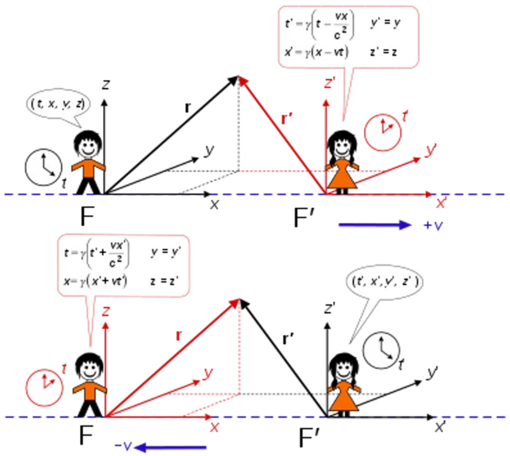
…….some possibilities in modifying gravity

- Relax some assumptions of GR:

  - Equivalence principle

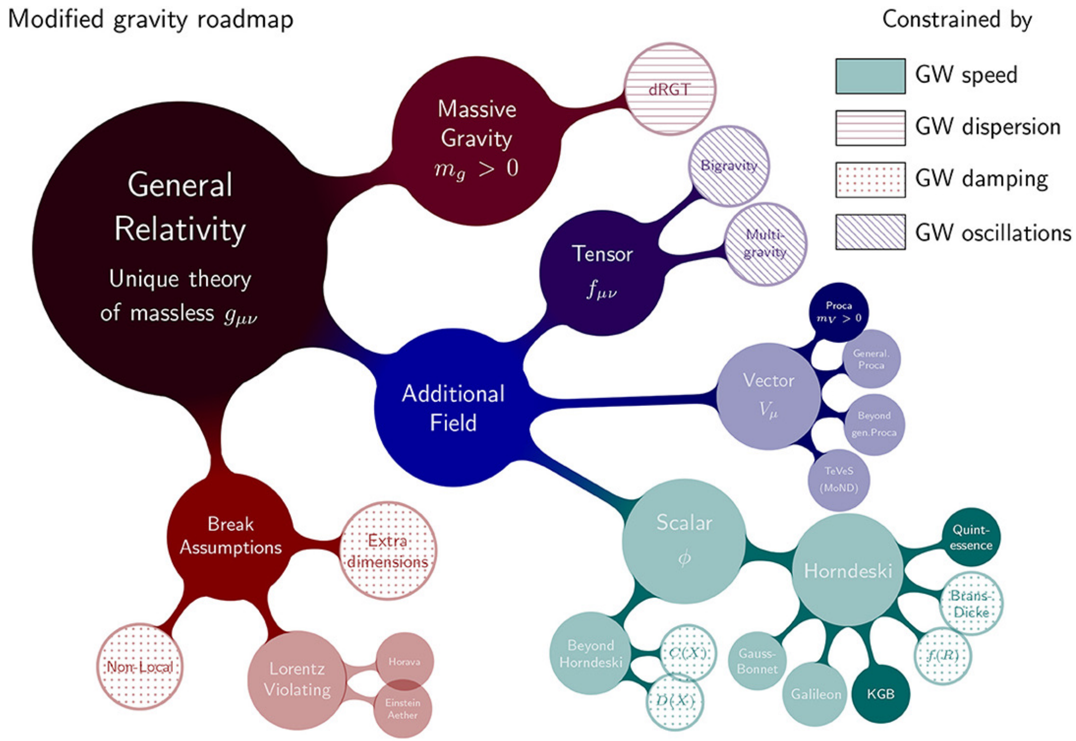


  - Lorentz invariance



  - Second-order field equations

$$S = \int \sqrt{-g} F(\phi, R, \Box^z R, R^{\mu\nu} R_{\mu\nu}, R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}) \qquad z \in \mathbb{Z}$$



Examples of *Local* Extended Theories of Gravity (ETGs)

…extended because we have to recover in someway GR

- Scalar-tensor Theories

$$S_{BD} = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\left[\phi R - \frac{\omega}{\phi} g^{\mu\nu}\,\nabla_\mu\,\phi\,\nabla_\nu\,\phi\right] + S^{(m)}$$

- Higher-order Theories

$$S_{Starobinsky} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,[R + \alpha R^2]\; + S^{(m)}$$

$$S_{Stelle} = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,[R + \alpha R^2 + \beta\, R^{\mu\nu} R_{\mu\nu}]\; + S^{(m)}$$

- Higher-order-scalar-tensor Theories

$$S = \int d^4x\,\sqrt{-g}\,\left[F(R, R, \Box^2 R, \ldots, \Box^k R, \phi)\; - \frac{\varepsilon}{2} g^{\mu\nu}\,\nabla_\mu\,\phi\,\nabla_\nu\,\phi\,\right] + 2\kappa\, S^{(m)}$$

Non-local ETGs

- Infinite Derivative Theories of Gravity (**IDG**s)

$$S \propto F_i(\Box_s)\, R$$

$$F_i(\Box_s) = \sum_{n=0}^{\infty} f_{i,n}\, \Box_s^n$$

- Integral Kernel Theories of Gravity (**IKG**s)

$$S \propto F(R, \Box^{-1} R)$$

$$S \propto F(T, \Box^{-1} T)$$

$$S \propto F(G, \Box^{-1} G)$$

They could be very useful to address astrophysical and cosmological scales and, eventually, infrared dynamics

R, T, G are geometric invariants (Curvature, Torsion, Gauss-Bonnet)

Infinite Derivative Theories of Gravity (**IDG**s)

We can start from the infinite-derivative Lorentz-invariant action depending on a scalar field

$$S = \frac{1}{2}\int d^4x d^4y\, \phi(x)\mathcal{K}(x-y)\phi(y) - \int d^4x V(\phi)$$

Prototype of Non-Locality:
a general operator depending on
the distance (x-y)

Starting from *S* and ***performing***:

1. A Fourier transformation
2. The reparameterization $\mathcal{K}(x-y) = F(\Box)\delta^{(4)}(x-y)$ with $F(\Box) = e^{-\gamma(\Box)}\prod_{i=1}^{N}(\Box - m_i^2)$

We get

$$\frac{1}{2}\int d^4x d^4y\, \phi(x)\mathcal{K}(x-y)\phi(y) \sim \frac{1}{2}\int d^4x \phi(x)F(\Box)\phi(x)$$

Infinite Derivative Theories of Gravity (**IDG**s)

The most general gravitational action in 4D, quadratic in curvature and ghost-free, has to contain infinite covariant derivatives:

$$S = \kappa \int d^4x \sqrt{-g} \left[R + \alpha \left(R F_1(\Box_s) R + R_{\mu\nu} F_2(\Box_s)\, R^{\mu\nu} + R_{\mu\nu\rho\sigma} F_3(\Box_s) R^{\mu\nu\rho\sigma}\right)\right] + S^{(m)}$$

- $\kappa \equiv (16\pi G_N)^{-1}$, $\alpha \equiv (M_s)^{-2}$, $[M_s] = length$
- $\Box_s \equiv \Box / M_s^2$, $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$
- $F_i(\Box_s)$ *transcendental and analytic* ⟶ $F_i(\Box_s) = \sum_{n=0}^{\infty} f_{i,n} \Box_s^n$

T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar. “Towards singularity and ghost free theories of gravity”. In: Phys. Rev. Lett. **108** (2012), p. 031101

Infinite Derivative Theories of Gravity (**IDG**s)

Super-renormalizable and Unitary theories

$$\mathcal{S} = \kappa \int d^4x \sqrt{-g} \left(R - G_{\mu\nu} \frac{e^{H(-\Box_s)} - 1}{\Box} R^{\mu\nu} \right)$$

$$\mathcal{S} = \kappa \int d^4x \sqrt{-g} \left[R - G_{\mu\nu} \frac{V_2^{-1} - 1}{\Box} R^{\mu\nu} + \frac{1}{2} R \frac{V_0^{-1} - V_2^{-1}}{\Box} R \right]$$

$$V_2^{-1} \equiv e^{H_2(-\Box_s)} p^{(n_2)}(-\Box_s) \,, \quad V_0^{-1} - V_2^{-1} \equiv \frac{1}{3} \left[e^{H_0(-\Box_s)} (1 + \Box_s) - e^{H_2(-\Box_s)} \right]$$

admit regular blackhole solutions

"maximal" UV-completion of $S_{Starobinsky}$

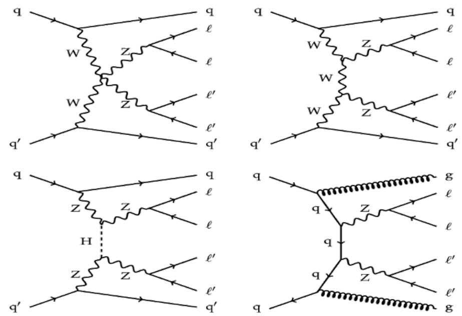
L. Modesto. "Super-renormalizable Quantum Gravity". In: Phys. Rev. **D86** (2012), p. 044005; F. Briscese, L. Modesto, and S. Tsujikawa. "Super-renormalizable or finite completion of the Starobinsky theory". In: Phys. Rev. **D89**.2(2014), p. 024029.

A possible classification of NLG models can come from Noether Symmetries

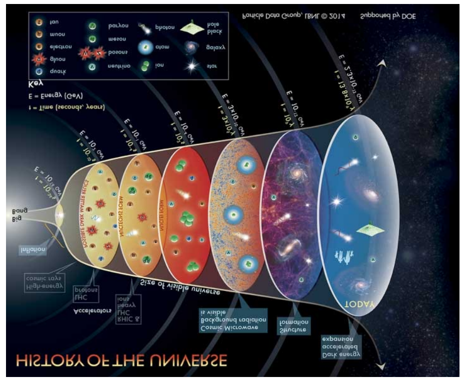
- Higher-order **IKG** (in the metric, affine, teleparallel formalism)
- Non-local extension of $\mathrm{f}(R) - gravity$

Motivations

- It could account for UV and IR quantum corrections →



- It could reproduce both UV and IR cosmic evolution →



Purposes

- Cosmography, Dark Energy
- Physically motivated cosmological models
- Reproducing cosmic history from UV to IF scales
- Search for NLG BH solutions where natural lengths are present

A possible method? → [*Noether Symmetry Approach*]

Noether Point Symmetries

$$\bar{t} = \bar{t}(t,q;\varepsilon) \simeq t + \varepsilon\xi(t,q)$$
$$\bar{q}^i = \bar{q}^i(t,q;\varepsilon) \simeq q^i + \varepsilon\eta^i(t,q)$$

→ *1-parameter (ε) group of point transformations*

$$\boldsymbol{X} = \xi(t,q)\frac{\partial}{\partial t} + \eta^i(t,q)\frac{\partial}{\partial q^i}$$

→ *infinitesimal group generator*

$$\boldsymbol{X}^{[1]} = \boldsymbol{X} + \eta^{[1]i}\frac{\partial}{\partial \dot{q}^i} = \boldsymbol{X} + \left(\dot{\eta}^i - \dot{\xi}\dot{q}^i\right)\frac{\partial}{\partial \dot{q}^i}$$

→ *"first prolongation" of the infinitesimal generator*

Noether Theorem. *If and only if it exists a function* $g(t,q(t))$ *such that*

$$\boldsymbol{X}^{[1]}L + \dot{\xi}L = \dot{g},$$

then the one-parameter group of point transformations generated by $\boldsymbol{X}$ *is a one-parameter group of Noether point symmetries for the dynamical system described by the Lagrangian* L.

The associated first integral of motion is:

$$I(t,q,\dot{q}) = \xi\left(\dot{q}\frac{\partial L}{\partial \dot{q}^i} - L\right) - \eta^i\frac{\partial L}{\partial \dot{q}^i} + g$$

Noether Symmetry Approach

The recipe:

1. Consider a class of point-like (cosmological, or spherically symmetric) Lagrangian

2. Write the ansatz for $\boldsymbol{X}$ ed $\boldsymbol{X}^{[1]}$

3. Derive the Noether point symmetry existence condition

$$\boldsymbol{X}^{[1]}L + \dot{\xi}L = \dot{g}$$

Obtain a polynomial depending on $\xi(t,q)$, $\eta^i(t,q)$, $\dot{g}(t,q)$ and products of the Lagrangian velocities $(e.g.\ \dot{\eta}^i\,\dot{\eta}^j\,\dot{\xi}\ \ldots)$ and a system of PDEs for $\xi, \eta^i, \dot{g}$

5. Select the form of Lagrangian

6. Solve, eventually, dynamics by first integrals.

The system contains the unknown function $F(R,\phi)$, so that it can provide, in principle, the explicit form for $F(R,\phi)$ related to the existence of symmetries. In other words, the existence of symmetries gives physically motivated Lagrangians. Φ represents NL terms.

Non-Local Gravity Cosmology

Based on:

S. Capozziello and F. Bajardi, ``Nonlocal gravity cosmology: An overview,'' Int. J. Mod. Phys. D **31** (2022) no.06, 2230009 doi:10.1142/S0218271822300099

A. Acunzo, F. Bajardi and S. Capozziello, ``Non-local curvature gravity cosmology via Noether symmetries,'' Phys. Lett. B **826** (2022), 136907 doi:10.1016/j.physletb.2022.136907

A first class of models has been proposed by Deser and Woodard

- ***Involve non-local operator of the form*** $\Box^{-1}$
- ***Deser and Woodard considered it in cosmology***

S. Deser and R. P. Woodard. "Nonlocal Cosmology". Phys. Rev. Lett. ***99*** *(2007), p. 111301*

They started from

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,R[1+F(\Box^{-1}R)] + S^{(m)}$$

- $\Box \equiv g^{\mu\nu}\,\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right)$
- $(\Box^{-1}R)(x) \equiv \int d^4x'\sqrt{-g}\,G(x,x')\,R(x')$ $\quad with \quad G(x,x')$ "retarded" Green

$\Box^{-1}$ *could explain the current late-time accelerated cosmic expansion without invoking any Dark Energy:*

$$g_{\mu\nu}^{FLRW} = diag(1, -a^2(t), -a^2(t), -a^2(t))$$

$$(\Box^{-1}R)(t) = \int_{t_i}^{t} dt' \,\frac{1}{a^3(t')}\int_{t_i}^{t'} dt'' a^3(t'')\,R(t'')$$

$t_i = t_{eq} \sim 10^5 y$

$t = t_0 \sim 10^{10} y$

$a(t) \sim t^s$

$s = 2/3$

The claim is : Current cosmic acceleration is recovered without any fine-tuning of parameters

Deser-Woodard model: cosmic acceleration

Enables a delayed response to the radiation-matter transition which could explain the current cosmic acceleration

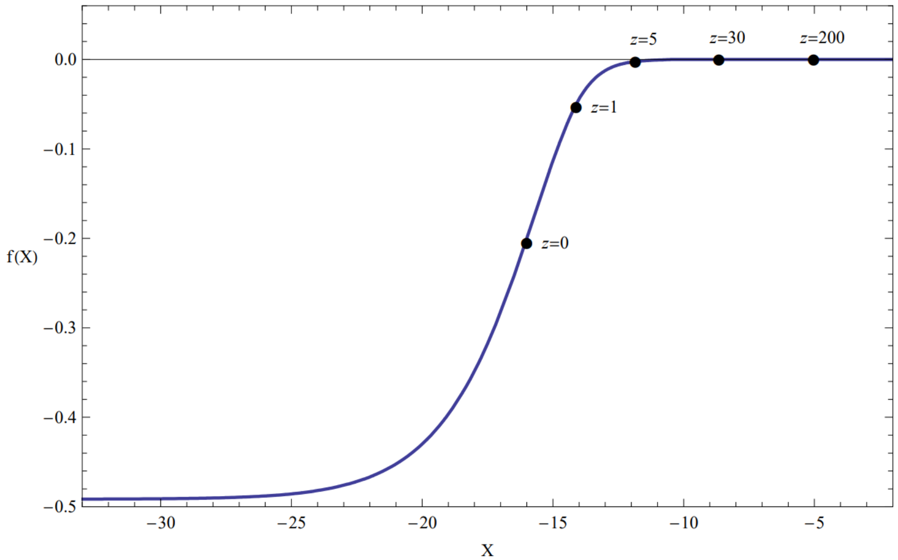
The steps:

- FLRW $\longrightarrow$ $\frac{1}{a^3(t)}\partial_t[a^3(t)\,\partial_t]$

$$[\Box^{-1}R](t) = G[R](t) = \int_0^t dt'\,\frac{1}{a^3(t')}\int_0^{t'} dt'' a^3(t'')\,R(t'')$$

$a(t) \;=\; t^{\frac{2}{3(1+\gamma)}}$

$RD\colon \gamma = \frac{1}{3}$ $\quad MD\colon \gamma = 0$



- $G[R](t)$ vanishes for $t = t_{equiv.}$
- $G[R](t)$ starts to grow for $t > t_{equ}$
- $G[R](t) \cong -14$ for $t = t_0{\sim}10^{10}\ yr$

Localization of Deser-Woodard model

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,R[1 + f(\Box^{-1}R)]$$

Scalar-tensor equivalent

$$\begin{cases} G_{\mu\nu} + \Delta G_{\mu\nu} = \kappa\, T_{\mu\nu}^{(m)} \\ \Delta G_{\mu\nu} = \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu\nu}\nabla_{\mu\nu}\right)\{f(\Box^{-1}R) + \Box^{-1}[R\, f'(\Box^{-1}R)]\} + \\ \left[\frac{1}{2}\left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha\right) - \frac{1}{2} g_{\mu\nu}\, g^{\alpha\beta}\right] \partial_\alpha(\Box^{-1}R)\partial_\beta\{\Box^{-1}[R\, f'(\Box^{-1}R)]\} \end{cases}$$

$$\eta(x) = \Box^{-1}R(x) \quad \Longrightarrow \quad R = \Box\eta$$

$$\mathcal{L} = \frac{1}{2\kappa}\{R[1 + f(\eta)] - \lambda(R - \Box\,\eta)\}$$

$$S = \frac{1}{2\kappa}\int d^4x\,\sqrt{-g}\,\{R[1 + f(\eta)] - \partial_\mu\xi\,\partial^\mu\eta - \xi R\} + S^{(m)}$$

$$\Box\eta = R\,, \quad \Box\xi = -R\,\frac{\partial f(\eta)}{\partial\eta}\,, \;\; G_{\mu\nu} = \kappa\, T_{\mu\nu}^{(m)} + \frac{1}{\Delta G_{\mu\nu}(\eta,\xi)}$$

S. Nojiri and S. D. Odintsov, "Modified non-local-f(r) gravity as the key for the inflation and dark energy," *Physics Letters B*, vol. 659, no. 4, 821–826, 2008, ISSN: 0370-2693. DOI: 10.1016/j.physletb.2007.12.001

Extension to general Lagrangians

$$\mathcal{S} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\,F(R,\Box^{-1}R)$$

$$S = \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,F(R,\phi)$$

formal localization

$$\phi \equiv \Box^{-1}R \quad\longrightarrow\quad R \equiv \Box\phi$$

$$g_{\mu\nu}^{FLRW} \Rightarrow \begin{cases} R = -6\left[\dfrac{\ddot{a}}{a} + \left(\dfrac{\dot{a}}{a}\right)^2\right] \\ R = \Box\phi = \ddot{\phi} + 3H\dot{\phi} \end{cases}$$

$$S = \kappa\int dt\,a^3\left\{F(R,\phi) - \epsilon(R - \ddot{\phi} - 3H\dot{\phi}) - \left(\frac{\partial F(R,\phi)}{\partial R} - \epsilon\right)\left[R + 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right]\right\}$$

$$L = a^3F - a^3\dot{\phi}\dot{\epsilon} - a^3R\,\partial_R F + 6a\dot{a}^2\partial_R F - 6a\dot{a}^2\epsilon + 6a^2\dot{a}\dot{R}\,\partial_{RR}F + 6a^2\dot{a}\dot{\phi}\,\partial_{R\phi}F - 6a^2\dot{a}\dot{\epsilon}$$

Minisuperspace

$$\boldsymbol{q}(t) = \{a(t), R(t), \phi(t), \epsilon(t)\}$$

New scalar field

Selection of the models by Noether symmetries

Noether Vector

$$X^{[1]} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial R} + \gamma\frac{\partial}{\partial \phi} + \delta\frac{\partial}{\partial \epsilon} + (\dot{\alpha} - \dot{\xi}\dot{a})\frac{\partial}{\partial \dot{a}} + (\dot{\beta} - \dot{\xi}\dot{R})\frac{\partial}{\partial \dot{R}} + (\dot{\gamma} - \dot{\xi}\dot{\phi})\frac{\partial}{\partial \dot{\phi}} + (\dot{\delta} - \dot{\xi}\dot{\epsilon})\frac{\partial}{\partial \dot{\epsilon}}$$

$$L = a^3 F - a^3\dot{\phi}\dot{\epsilon} - a^3 R\,\partial_R F + 6a\dot{a}^2\partial_R F - 6a\dot{a}^2\epsilon + 6a^2\dot{a}\dot{R}\,\partial_{RR}F + 6a^2\dot{a}\dot{\phi}\,\partial_{R\phi}F - 6a^2\dot{a}\dot{\epsilon}$$

System of 28 PDE

2 classes of solutions: same generator, different functions

$$\mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \frac{\xi_0}{3}(2n-1)\partial_a - 2\xi_0 R\partial_R + \frac{2\xi_0(1-\ell)}{n}\partial_\phi + (2\xi_0(1-n)\epsilon + \delta_1)\partial_\epsilon$$

$$f_I(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + [2\xi_0 R]^n\mathcal{F}\left(\phi + \frac{(1-n)}{\ell}\log[2\xi_0 R]\right)$$

$$f_{II}(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + G(R)\,e^{k\phi}$$

Two interesting cases

First Case

A possible choice

$$\mathcal{F}_1\left(\phi + \frac{(1-n)}{\ell}\log[2\xi_0 R]\right) \equiv \phi + \frac{(1-n)}{\ell}\log[2\xi_0 R] + q$$

The function becomes

$$f_1(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + (2\xi_0 R)^n(q+\phi) + (2\xi_0 R)^n\,\frac{(1-n)}{\ell}\log[2\xi_0 R]$$

Example for n=2 → **NON-LOCAL EXTENSION OF STAROBINSKY MODEL RECOVERED BY NOETHER SYMMETRIES**

$$f_1(R,\phi)\Big|_{n=2} = \frac{\delta_1}{2\xi_0(n-1)}R + 4\,\xi_0^2 R^2(q+\phi) - \frac{4\xi_0^2}{\ell}R^2\,\log[2\xi_0 R]$$

Cosmological Solutions for the First Case

Replacing $f_1(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)} R + (2\xi_0 R)^n (q+\phi) + (2\xi_0 R)^n \frac{(1-n)}{\ell} \log[2\xi_0 R]$

Into the system of E-L equations, we get different exact cosmological solutions e.g.

I: $a(t) = a_0\, e^{\Lambda t} \quad R(t) = -12\,\Lambda^2 \quad \phi(t) = -\frac{1}{3}(40+3q) - 4\Lambda t$

$$\epsilon(t) = 576(2\xi_0)^3 \Lambda^5 t - \frac{C_3 e^{-3\Lambda t}}{3\Lambda} + \frac{\delta_1}{2\xi_0(n-1)},$$

II: $a(t) = a_0\, t^{-10} \quad R(t) = -1260\, t^{-2} \quad \phi(t) = C_2 + \frac{1260}{31}\log(t)$

$$\epsilon(t) = \frac{\delta_1}{2\xi_0(n-1)} + \frac{C_3}{31} t^{31} + 14288400(2\xi_0)^3\, t^{-4}$$

Non-locality can be easily restored by $\phi \equiv \Box^{-1} R$

Cosmological Solutions for the First Case

$$f_1(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + (2\xi_0 R)^n(q+\phi) + (2\xi_0 R)^n \frac{(1-n)}{\ell}\log[2\xi_0 R]$$

Exact radiation solutions:

$$a(t) = a_0\, t^{\frac{1}{2}} \quad R(t) = 0 \quad \phi(t) = C_2 \quad \epsilon(t) = \frac{\delta_1}{2\xi_0(n-1)} - \frac{2C_3}{\sqrt{t}}$$

A possible case is

$$f_1(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + \phi \qquad\qquad \phi \equiv \Box^{-1}R$$

and then the minimal Deser-Woodard case is easily recovered

Cosmological Solutions for the Second Case

$$f_{II}(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)}R + G(R)\,e^{k\phi}$$

⬇

$$a(t) = a_0 e^{mt} \quad \phi(t) = -4h_0 mt \quad R(t) = -12m^2$$

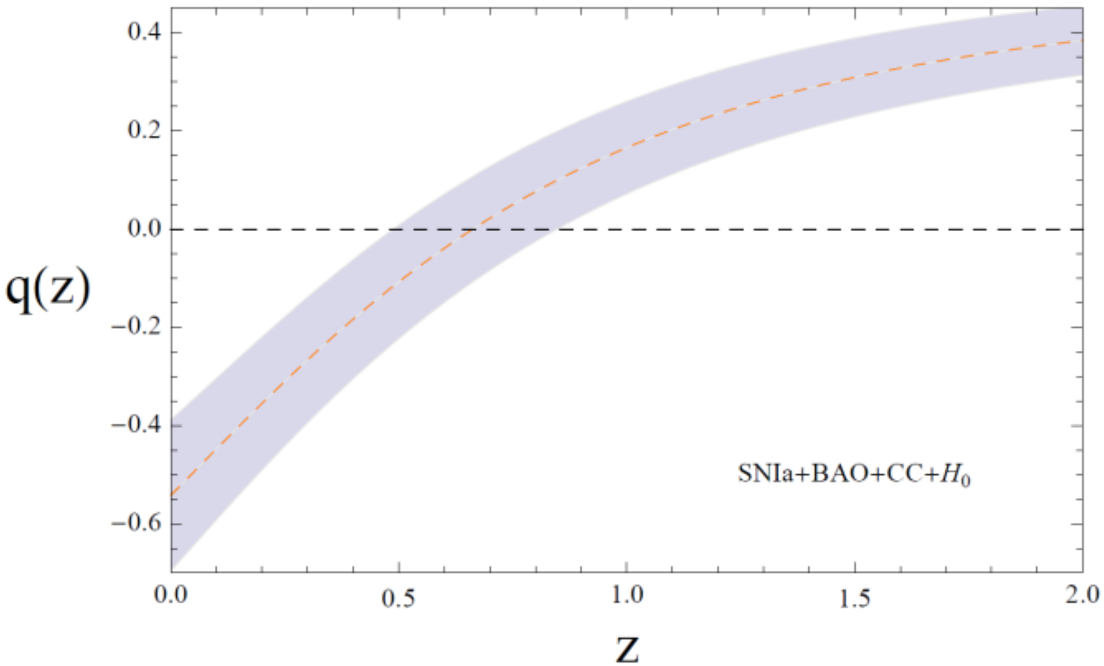
$$\epsilon(t) = \frac{e^{-3mt}\left[\frac{3^{1+n}4^n e^{(3-4h_0k)mt} f_0(-m^2)^n}{h_0(3-4h0k)}\right]}{(12m^2)}$$

This case is interesting because it reproduces the above super-renormalizable model and gives rise to compatible dark-energy models.

L. Modesto. “Super-renormalizable Quantum Gravity”. In: Phys. Rev. **D86** (2012), p. 044005; F. Briscese, L. Modesto, and S. Tsujikawa. “Super-renormalizable or finite completion of the Starobinsky theory”. In: Phys. Rev. **D89**.2(2014), p. 024029.

Observational Perspectives

- Observational constrains of the model free parameters *via* cosmological data, *e.g.* $\mathrm{SNe\ Ia} + \mathrm{BAO} + \mathrm{CC} + H_0$
- Compatibility with PLANCK data
- Searching for new cosmological constraints for the non-local terms
- The Deser-Woodard model is a particular case of a wide class of models selected by Noether symmetries



*S. Bahamonde, S. Capozziello, M. Faizal, R. C. Nunes. "Nonlocal Teleparallel Cosmology". In: Eur. Phys. J. **C**77.9 (2017), p.628*

Astrophysical tests by Galactic Centre

Based on:

K.F. Dialektopoulos, D. Borka, S. Capozziello, V. Borka Jovanovic, P. Jovanovic "Constraining non-local gravity by S2 star orbits". In: Phys. Rev. D **99** (2019), p. 044053

S. Capozziello, D. Borka, P. Jovanovic, V. Borka Jovanovic, "Constraining Extended Gravity Models by S2 star orbits around the Galactic Centre". In: Phys. Rev. D **90** (2014), p. 044052

Objectives

- Selecting non-local action in spherical symmetry by Noether symmetries

- Performing the post-Newtonian limit

- Constraining the free parameters by S2 star orbiting around SgrA*

- Estimate the reduced χ^2 and constrain characteristic lengths related to NLG

Non-Local Gravity in Spherical Symmetry

We focus on a spherically symmetric spacetime

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2 d\Omega^2$$

We use again Noether symmetries

$$\phi \equiv \Box^{-1} R \longrightarrow S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left\{R[1+f(\phi)] + \varepsilon(r,t)(\Box\phi - R)\right\}d^4x$$

New scalar field depends on both r and t

we generalize the ***Deser and Woodard model***

$$\mathcal{L}(r,\nu,\lambda) = e^{-\frac{1}{2}(\lambda+\nu)}\Big[-e^{\nu}r^2\nu_r\phi_r f_\phi(\phi) + e^{\lambda}r^2\lambda_t\phi_t f_\phi(\phi) + \\ -2e^{\nu}f(\phi)\left(e^{\lambda}+r\lambda_r-1\right) - 2e^{\lambda+\nu} + 2e^{\nu} + e^{\nu}r^2\varepsilon_r\phi_r + e^{\nu}r^2\nu_r\varepsilon_r + \\ -e^{\lambda}r^2\varepsilon_t\phi_t - e^{\lambda}r^2\lambda_t\varepsilon_t + 2e^{\nu}\varepsilon\left(e^{\lambda}+r\lambda_r-1\right) - 2e^{\nu}r\lambda_r\Big]$$

Solutions

Noether symmetries select

$$\begin{cases} \mathcal{X} = (\xi_0 t + \xi^t(r))\partial_t - 2\xi_0\partial_\nu + (\gamma_0 + 2\xi_0)\partial_\phi + \delta_0(\gamma_0 + 2\xi_0)\partial_\varepsilon \\ f(\phi) = \delta_0\phi + f_1 \\ \\ \mathcal{X} = (\xi_0 t + \xi^r(r))\partial_t - \frac{\xi_1}{2} r\partial_r - (2\xi_0 + \xi_1)\partial_\nu + \gamma_0\partial_\phi + \xi_1(\varepsilon - \delta_0 - 1)\partial_\varepsilon \\ f(\phi) = \delta_0 + f_1 e^{\frac{\gamma_0}{\xi_1}\phi} \end{cases}$$

$\xi^\mu = (\xi^t, \xi^r, 0, 0)$

1) We restrict the interval to a subclass of spacetimes of the form

$$ds^2 = A(r)dt^2 - B(r)\,dr^2 - r^2 d\Omega^2$$

2) We consider up to sixth-order approximation of the metric

$$g_{00} \sim \mathcal{O}(6),\, g_{0i} \sim \mathcal{O}(5) \text{ and } g_{ij} \sim \mathcal{O}(4)$$

Post Newtonian Limit

The approximation $g_{00} \sim \mathcal{O}(6), g_{0i} \sim \mathcal{O}(5)$ and $g_{ij} \sim \mathcal{O}(4)$

Potentials

$$\begin{cases} A(r) = 1 + \frac{1}{c^2}\Phi(r)^{(2)} + \frac{1}{c^4}\Phi(r)^{(4)} + \frac{1}{c^6}\Phi(r)^{(6)} + \mathcal{O}(8) \\ B(r) = 1 + \frac{1}{c^2}\Psi(r)^{(2)} + \frac{1}{c^4}\Psi(r)^{(4)} + \mathcal{O}(6) \\ \phi(r) = \phi_0 + \frac{1}{c^2}\phi(r)^{(2)} + \frac{1}{c^4}\phi(r)^{(4)} + \frac{1}{c^6}\phi(r)^{(6)} + \mathcal{O}(8) \\ \varepsilon(r) = \varepsilon_0 + \frac{1}{c^2}\varepsilon(r)^{(2)} + \frac{1}{c^4}\varepsilon(r)^{(4)} + \frac{1}{c^6}\varepsilon(r)^{(6)} + \mathcal{O}(8) \end{cases}$$

The above functions can be replaced into the field equations

$$[1+f(\phi)-\varepsilon]G_{\mu\nu} = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\ f(\phi) - \frac{1}{2}g_{\mu\nu}D_\alpha\varepsilon D^\alpha\phi + D_\mu\varepsilon D_\nu\phi$$

Corrected Newtonian potentials

Replacing the second function selected by Noether's approach

$$f(\phi) = \delta_0 + f_1 e^{\frac{\gamma_0}{\xi_1}\phi}$$

into the field equations, with the approximations

$$\begin{cases} A(r) = 1 + \frac{1}{c^2}\Phi(r)^{(2)} + \frac{1}{c^4}\Phi(r)^{(4)} + \frac{1}{c^6}\Phi(r)^{(6)} + \mathcal{O}(8) \\ B(r) = 1 + \frac{1}{c^2}\Psi(r)^{(2)} + \frac{1}{c^4}\Psi(r)^{(4)} + \mathcal{O}(6) \\ \phi(r) = \phi_0 + \frac{1}{c^2}\phi(r)^{(2)} + \frac{1}{c^4}\phi(r)^{(4)} + \frac{1}{c^6}\phi(r)^{(6)} + \mathcal{O}(8) \\ \varepsilon(r) = \varepsilon_0 + \frac{1}{c^2}\varepsilon(r)^{(2)} + \frac{1}{c^4}\varepsilon(r)^{(4)} + \frac{1}{c^6}\varepsilon(r)^{(6)} + \mathcal{O}(8) \end{cases}$$

We obtain

$$A(r) = 1 - \frac{2G_N M\phi_c}{c^2 r} + \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{14}{9}\phi_c^2 + \frac{18r_\varepsilon - 11r_\phi}{6r_\varepsilon r_\phi}r\right] + - \frac{G_N^3 M^3}{c^6 r^3}\left[\frac{50r_\varepsilon - 7r_\phi}{12r_\varepsilon r_\phi}\phi_c r + \frac{16\phi_c^3}{27} - \frac{r^2\left(2r_\varepsilon^2 - r_\phi^2\right)}{r_\varepsilon^2 r_\phi^2}\right]$$

$$B(r) = 1 + \frac{2G_N M\phi_c}{3c^2 r} + \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{2\phi_c^2}{9} + \left(\frac{3}{2r_\varepsilon} - \frac{1}{r_\phi}\right)r\right]$$

$$\phi(r) = \frac{4G_N M\phi_c}{3c^2 r} - \frac{G_N^2 M^2}{c^4 r^2}\left[\left(\frac{11}{6r_\varepsilon} + \frac{1}{r_\phi}\right)r - \frac{2\phi_c^2}{9}\right] + - \frac{G_N^3 M^3}{c^6 r^3}\left[\frac{r^2}{r_\phi^2} - \left(\frac{25}{12r_\varepsilon} - \frac{7}{6r_\phi}\right)\phi_c r - \frac{4\phi_c^3}{81}\right]$$

$$\varepsilon(r) = 1 + \frac{G_N^2 M^2}{c^4 r^2}\left[\frac{2\phi_c^2}{3} - \left(\frac{13}{6r_\varepsilon} - \frac{1}{r_\phi}\right)r\right] + + \frac{G_N^3 M^3}{c^6 r^3}\left[\frac{20\phi_c^3}{27} - \left(\frac{1}{r_\varepsilon^2} - \frac{1}{r_\phi^2}\right)r^2 - \left(\frac{131}{36r_\varepsilon} + \frac{1}{6r_\phi}\right)\phi_c r\right]$$

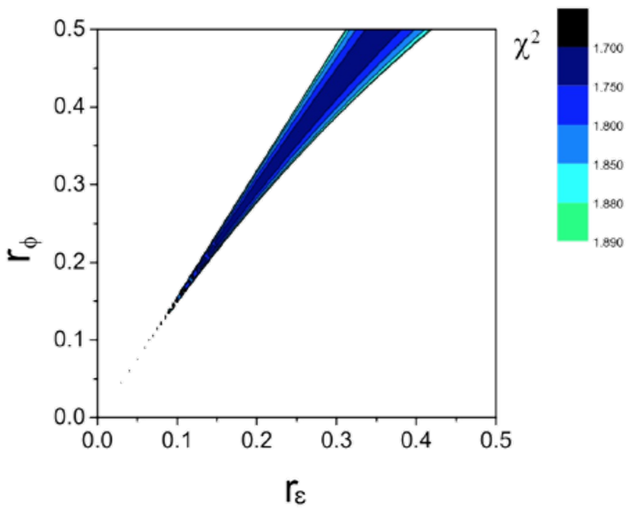
$$\Phi^{(2)}(r) = -\frac{2G_N M}{r}\phi_c$$

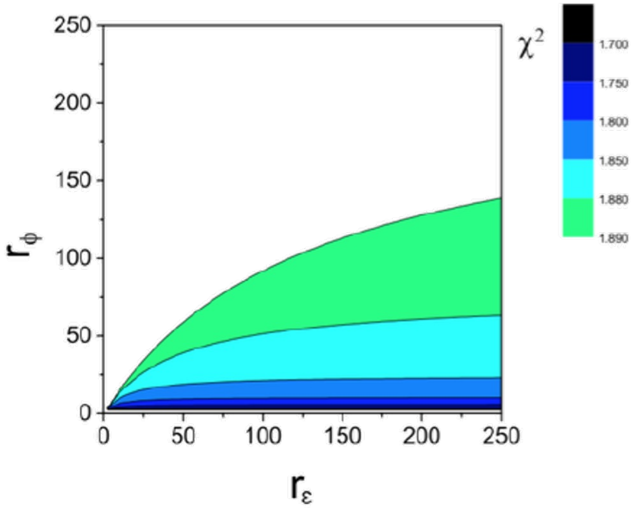
$$\Phi^{(4)}(r) = \frac{G_N^2 M^2}{r^2}\left[\frac{14}{9}\phi_c^2 + \frac{18r_\varepsilon - 11r_\phi}{6r_\varepsilon r_\phi}r\right]$$

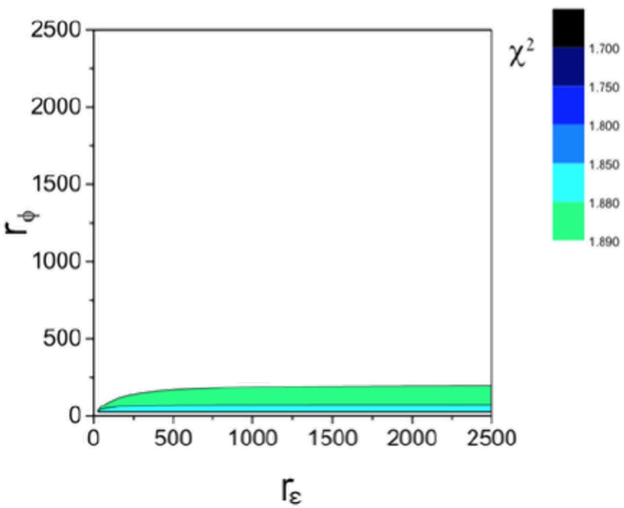
$$\Phi^{(6)}(r) = \frac{G_N^3 M^3}{r^3}\left[\frac{7r_\phi - 50r_\varepsilon}{12r_\varepsilon r_\phi}\phi_c r - \frac{16\phi_c^3}{27} + \frac{2r_\varepsilon^2 - r_\phi^2}{r_\varepsilon^2 r_\phi^2}r^2\right]$$

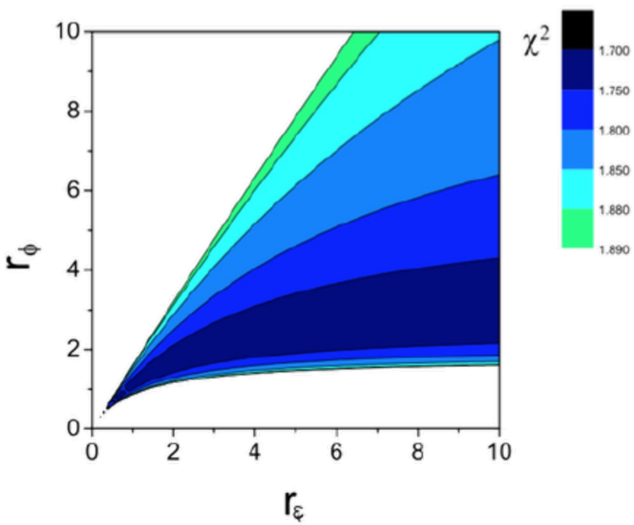
Order of the potential

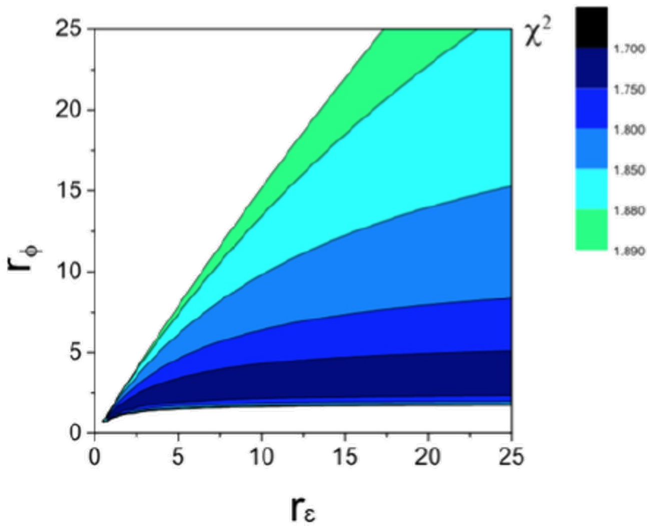
Two new length appears: r_ε and r_ϕ, searching for those by simulated orbits giving *at least* the same χ^2 as the Keplerian orbit $(\chi^2 \sim 1.89)$











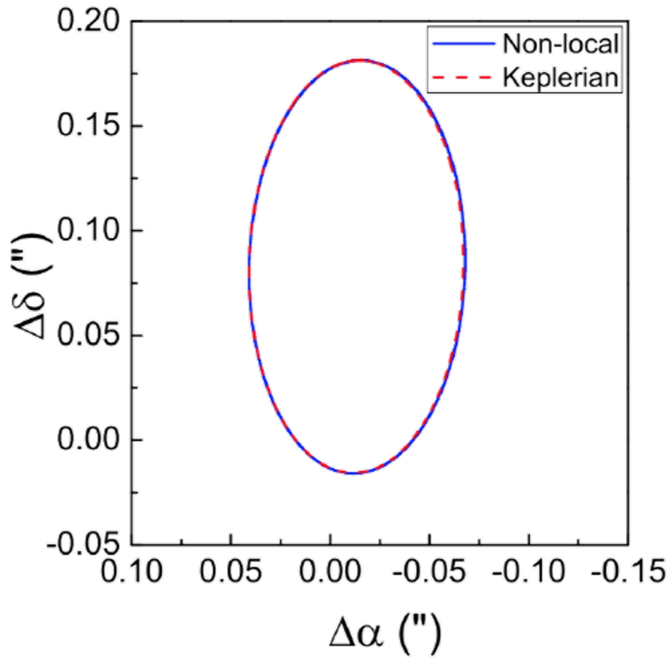
Maps of the reduced χ^2 in different regions, representing the lengths in AU giving at least the same χ^2 as the Keplerian orbit

After fixing the right parameters minimizing the χ^2 we plot the orbit

Comparisons between the Keplerian orbit of S2 star (red dashed line) and the orbit predicted by Non-Local gravity (blue solid line) with parameter values that minimize the χ^2 :

$$r_\phi \sim 1.2\, AU \; and \; r_\varepsilon \sim 1.1\, AU.$$

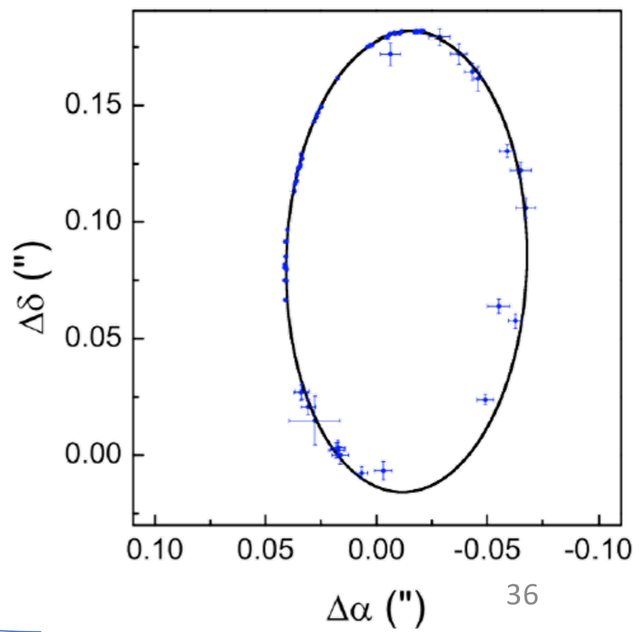
compatible with the EHT measurements!



$\Delta\alpha \; and \; \Delta\delta$

coordinates of S2 star

Same comparison but with the error bars. Same value for the characteristic lengths.



Constraining Non − Local Gravity by Clusters of Galaxies

Based on:

S. Capozziello, M. Faizal, M. Hameeda, B. Pourhassan and V. Salzano, ``*Logarithmic corrections to Newtonian gravity and Large Scale Structure*,'' Eur. Phys. J. C **81** (2021) no.4, 352

F. Bouchè, S. Capozziello, V. Salzano and K. Umetsu, ``Testing non-local gravity by clusters of galaxies,'' Eur. Phys. J. C**82** (2022) 7, 652

Again weak field approximation

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2$$

- Spherically symmetric metric
- Birkhoff's theorem as a good approximation in the PN limit
- Solution $\mathrm{B}(r) = 1/A(r)$ not guaranteed in nonlocal gravity

↓

Post-Newtonian limit

$$A(r) = 1 + \frac{1}{c^2}\phi^{(2)} + \frac{1}{c^4}\phi^{(4)} + \frac{1}{c^6}\phi^{(6)} + \mathcal{O}(8)$$

$$B(r) = 1 + \frac{1}{c^2}\psi^{(2)} + \frac{1}{c^4}\psi^{(4)} + \mathcal{O}(6)$$

$$\eta(r) = 1 + \frac{1}{c^2}\eta^{(2)} + \frac{1}{c^4}\eta^{(4)} + \frac{1}{c^6}\eta^{(6)} + \mathcal{O}(8)$$

$$\xi(r) = 1 + \frac{1}{c^2}\xi^{(2)} + \frac{1}{c^4}\xi^{(4)} + \frac{1}{c^6}\xi^{(6)} + \mathcal{O}(8)$$

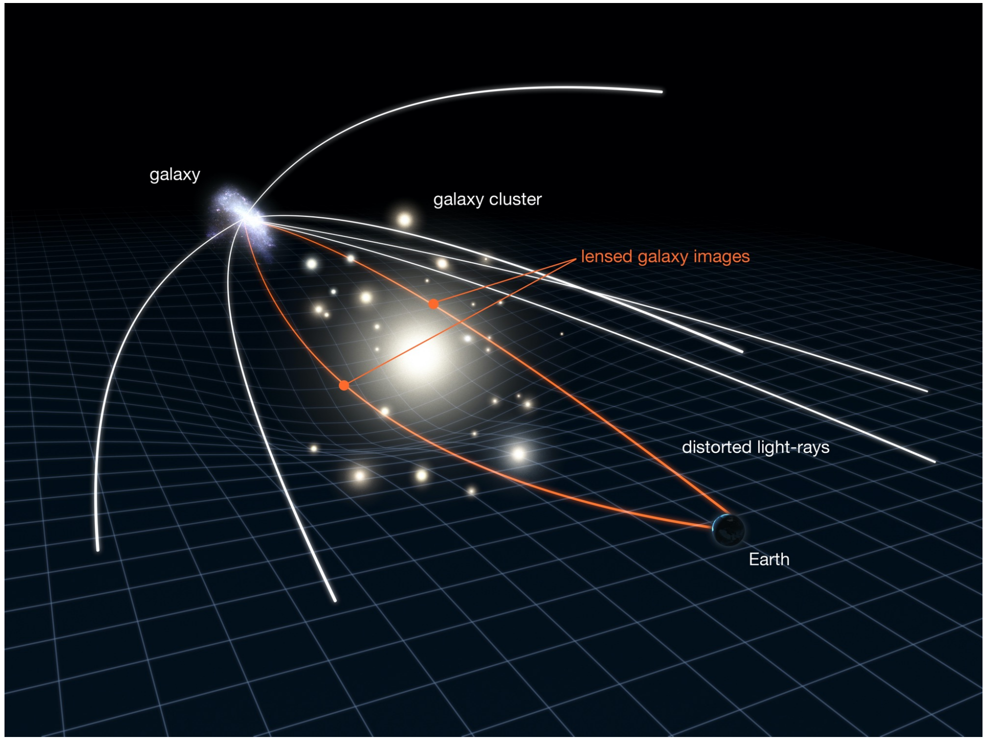
substituting into Klein-Gordon equations and ''0,0'' and ''1,1'' component of the gravitational field equations →

$$g_{00} = A(r) = 1 + \frac{2\Phi(r)}{c^2} \qquad g_{11} = -B(r) = 1 - \frac{2\Psi(r)}{c^2}$$

$$\Phi(r) = -\frac{GM}{r} + \frac{G^2M^2}{2c^2r^2}\left[\frac{14}{9} + \left(\frac{3}{r_\eta} - \frac{11}{6r_\xi}\right)r\right] + \frac{G^3M^3}{2c^4r^3}\left[\left(\frac{7}{12r_\xi} - \frac{25}{6r_\eta}\right)r - \frac{16}{27} + \left(\frac{2}{r_\eta^2} - \frac{1}{r_\xi^2}\right)r^2\right]$$

$$\Psi(r) = -\frac{GM}{3r} + \frac{G^2M^2}{2c^2r^2}\left[\frac{2}{9} + \left(\frac{3}{2r_\xi} - \frac{1}{r_\eta}\right)r\right]$$

Tests by Gravitational Lensing



We calculate the theoretical lensing convergence and compare the results with lensing data provided by *CLASH* (Cluster Lensing and Supernova survey with Hubble).

Lensing convergence: $\kappa(R) = \frac{1}{c^2}\frac{D_{ds}D_d}{D_s}\int_{-\infty}^{+\infty} dz\, \nabla_r^2 \left(\frac{\Phi(R,z)+\Psi(R,z)}{2}\right)$

Generalize the point-like potentials to extended, spherically symmetric mass distributions. We consider the previously selected model to test the NL contribution

$$f(\phi) = \delta_0 + f_1 e^{\frac{\gamma_0}{\xi_1}\phi}$$

$$\phi \equiv \Box^{-1} R$$

1. Estimate the orders of magnitude of each contribution

$\mathcal{O}\left(\frac{GM}{r}\right) \sim 10^{-27}\ \mathrm{kpc^2 s^{-2}}$ $\quad \mathcal{O}\left(\frac{G^2M^2}{2c^2r^2}\right) \sim 10^{-32}\ \mathrm{kpc^2 s^{-2}}$

$\mathcal{O}\left(\frac{G^3M^3}{2c^4r^3}\right) \sim 10^{-32}\ \mathrm{kpc^2 s^{-2}}$ → the third order can be neglected

2. Choose a mass density profile → NFW

$$\rho_{NFW}(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2} \qquad \begin{cases} \rho_s = \frac{\Delta}{3}\rho_{cr}\frac{c_\Delta^3}{\ln(1+c_\Delta) - \frac{c_\Delta}{1+c_\Delta}} \\ r_s = \frac{r_\Delta}{c_\Delta} \end{cases}$$

3. Extend the potentials by integrating over infinitesimal mass elements

- the integration over the radial coordinate r' has to be performed between 0 and r and between r and ∞, because Newton's theorems are not guaranteed in nonlocal gravity
- the mass element is $dM = \rho(r')\, r'^2 dr' sin\theta\, d\theta d\varphi$, while $M^2 \to 2MdM = 2\, dM(r') \int_0^{r'} dr''\, r''^2\, \rho(r'')$
- the terms $1/r$ and $1/r^2$ enter the integral as $1/|\boldsymbol{r}-\boldsymbol{r}'|$ and $1/|\boldsymbol{r}-\boldsymbol{r}'|^2$, where $|\boldsymbol{r}-\boldsymbol{r}'| = (r^2 + r'^2 - 2rr'cos\theta)^{\frac{1}{2}}$

Statistical analysis adopting the Deser-Woodard model

Data sets: taken from CLASH program

Free parameters

$$\boldsymbol{\theta} = \{c_{200}, M_{200}, r_\eta, r_\xi\} \longrightarrow \chi^2 = [\boldsymbol{\kappa}^{theo}(\boldsymbol{\theta}) - \boldsymbol{\kappa}^{obs}] \cdot \boldsymbol{C}^{-1} \cdot [\boldsymbol{\kappa}^{theo}(\boldsymbol{\theta}) - \boldsymbol{\kappa}^{obs}]$$

- $\boldsymbol{C}^{-1}$ → Inverse covariant matrix
- $\boldsymbol{\kappa}^{obs}$ → 15 elements vector with observed values
- $\boldsymbol{\kappa}^{theo}$ → Theoretical lensing convergence

$$\kappa(R) = \frac{1}{c^2}\frac{D_{ds}D_d}{D_s}\int_{-\infty}^{+\infty} dz\, \nabla_r^2 \left(\frac{\Phi(R,z) + \Psi(R,z)}{2}\right)$$

3-step procedure for the minimization

1. First preliminary MCMC, 10˙000 steps long, with arbitrary initial values and the following covariance matrix

$$\mathbf{C}_{\text{proposal}} = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix}$$

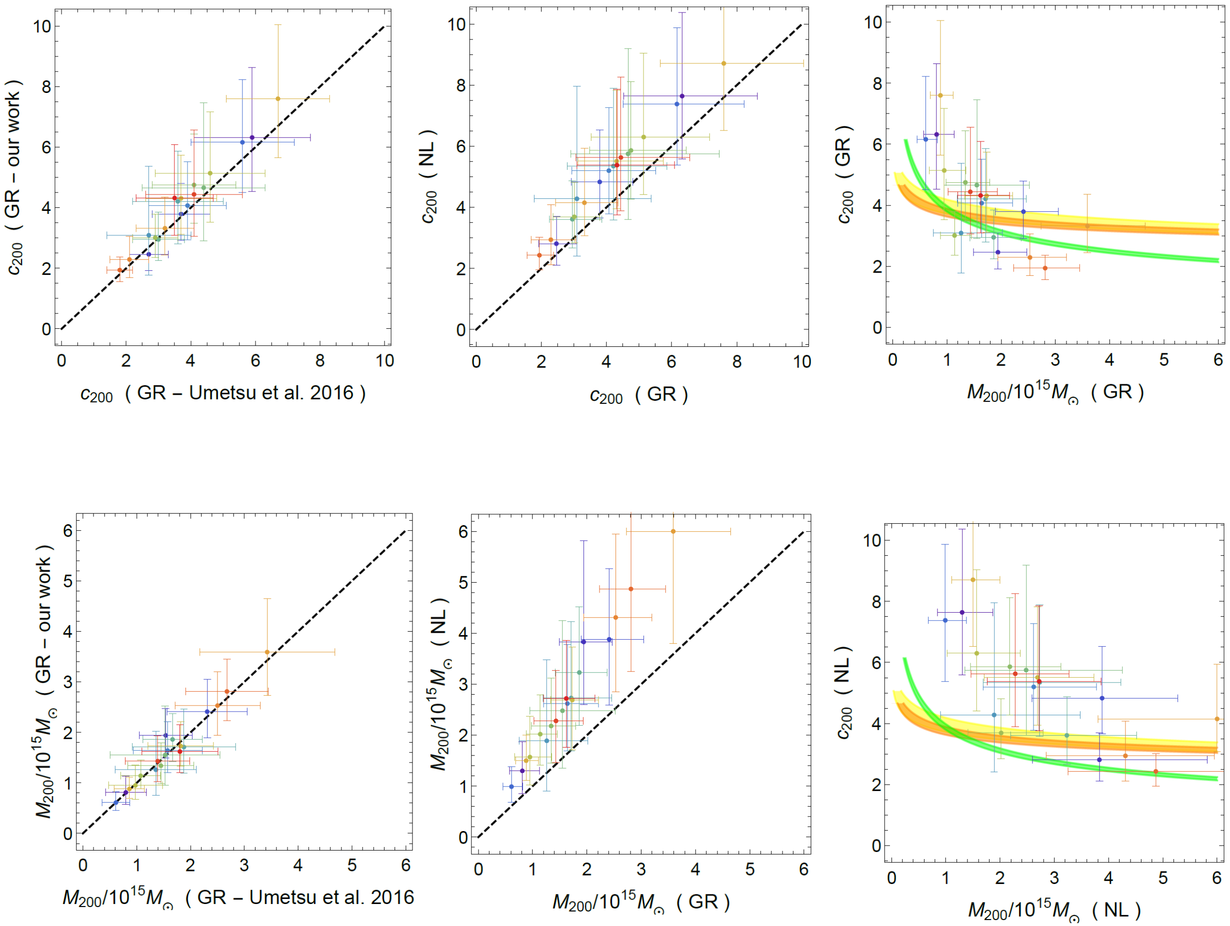
2. Second preliminary MCMC of 10˙000 iterations. As initial values and covariance matrix we used the ones resulting from the first MCMC (minimum of the χ^2)
3. Definitive Markov Chain of 50˙000 steps. As initial values and covariance matrix we used the ones resulting from the second MCMC (minimum of the χ^2)

Results: NFW parameters

Fit in the GR scenario: agreement within 1σ with results from literature

Fit in the NL scenario: shift towards higher values with respect to GR

Cross-check with c_{200}- M_{200} relations from literature: for GR the region spanned by the clusters agrees with the bands, while for NL, the region shifts towards higher concentrations and masses



M_{200} is used as the halo mass, which is the total mass contained within R_{200}, the radius within which the enclosed over-density is 200 times the critical density.

J. Merten *et al.*, arXiv: 1404.1376 [astro-ph.CO].

B. Diemer and M. Joyce, arXiv: 1809.07326 [astro-ph.CO].

C. A. Correa, J. S. B.Wyithe, J. Schaye, and A. R. Duffy, arXiv: 1502.00391 [astro-ph.CO].

Results for non-local length scales

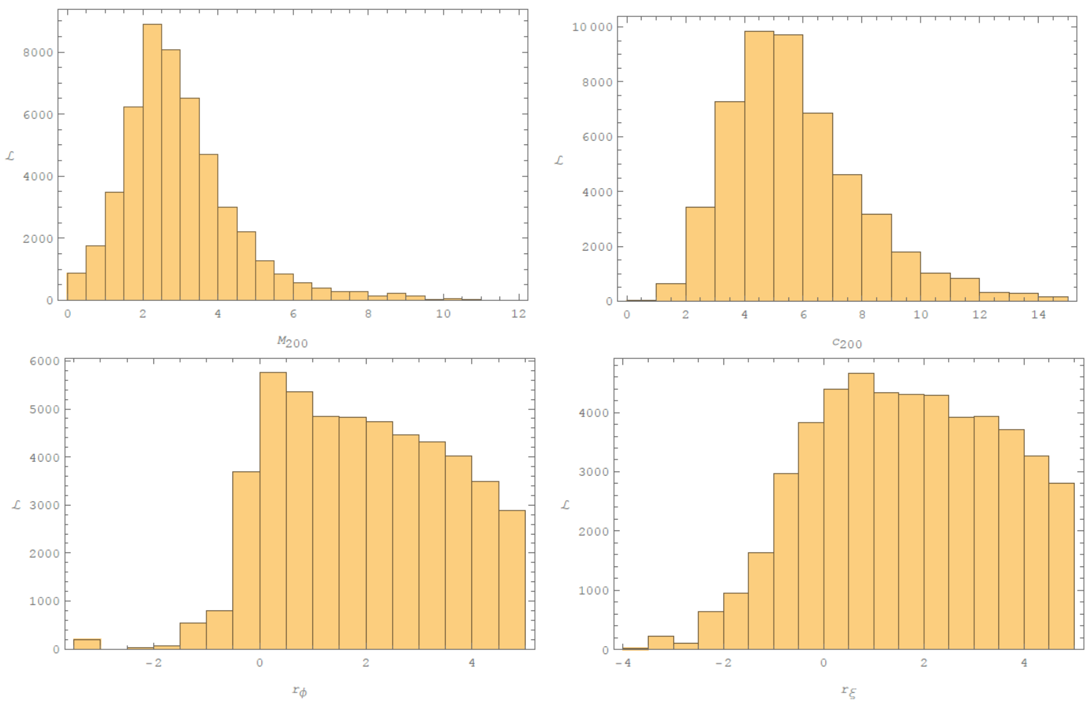
Typical lower bounds for the non-local parameters

$$r_\eta > 4 \cdot 10^{-5} - 7 \cdot 10^{-2}\mathrm{kpc} \qquad r_\xi > 2 \cdot 10^{-5} - 3 \cdot 10^{-2}\mathrm{kpc}$$

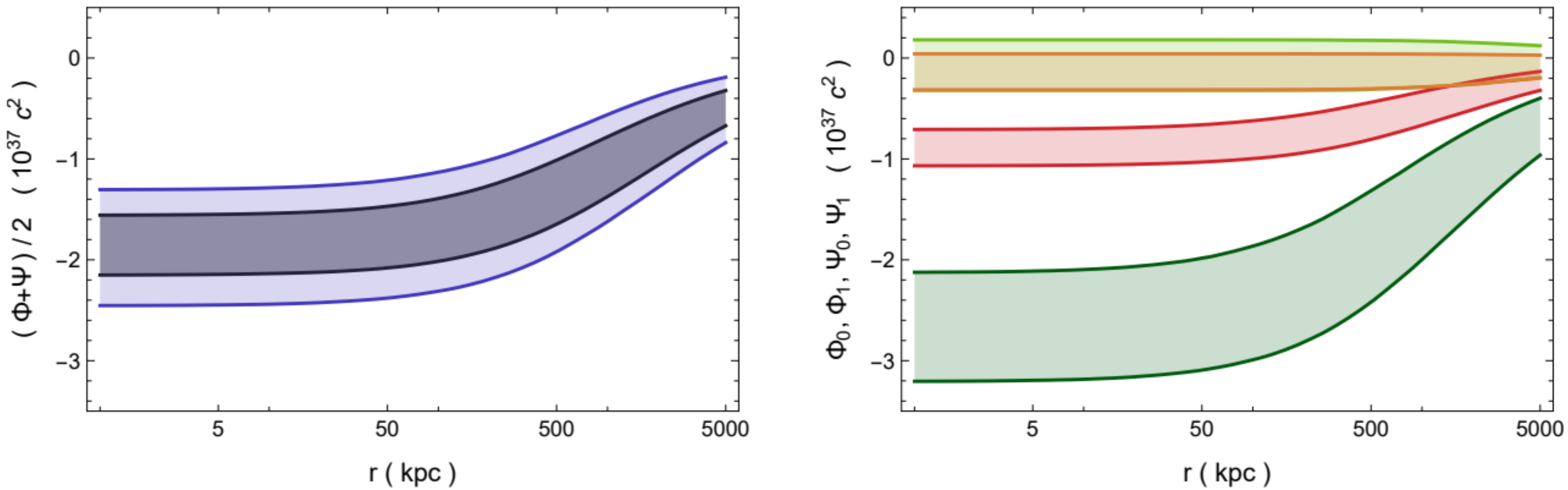
Corresponding magnitude of the non-local corrections to the potential

$$\Phi_1 \sim 10^{-28} - 10^{-25}\mathrm{kpc}^2\mathrm{s}^{-2} \qquad \Psi_1 \sim 10^{-27} - 10^{-24}\mathrm{kpc}^2\mathrm{s}^{-2}$$

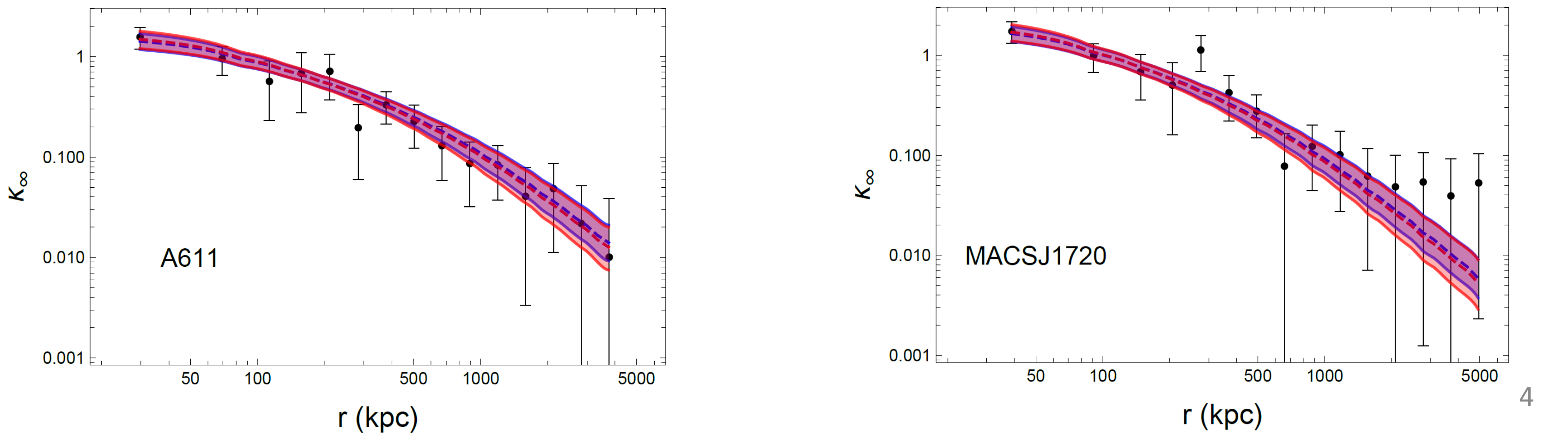
Statistics for typical clusters



New potentials occurring in Non-Local Gravity can be used to investigate the gravitational lensing



Gravitational lensing potential (dimensionless) contributions for A611. *Left panel*: total gravitational potential for GR (black) and nonlocal (blue). *Right panel*: $\Phi(r)$ (green) and $\Psi(r)$ (red) contributions; zeroth-order terms are in dark colors, first-order terms are in light color



Gravitational Waves in Non—Local Gravity

Based on:

S. Capozziello and M. Capriolo, ``*Gravitational waves in non-local gravity*,'' Class. Quant. Grav. **38** (2021) no.17, 175008

S. Capozziello, M. Capriolo, S. Nojiri ``*Consideration on Gravitational waves in higher-order local and non-local gravity*,'' Phys. Lett. B. **810** (2020), 135821

Let us start from one of the two functions containing symmetries, that is

$$f_1(R,\phi) = \frac{\delta_1}{2\xi_0(n-1)} R + (2\xi_0 R)^n (q+\phi) + (2\xi_0 R)^n \frac{(1-n)}{\ell} \log[2\xi_0 R]$$

Setting n = 1 and q = 0, the action can be recast as:

with $\phi \equiv \Box^{-1} R$

$$S[g] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R + a_1 R \Box^{-1} R\right)$$

or, in terms of Lagrange multipliers, as

$$S_g[g,\phi,\lambda] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R\left(1 + a_1\phi\right) + \lambda\left(\Box\phi - R\right)\right]$$

Plugging the first-order expansions →

$$\begin{cases} g_{\mu\nu} \sim \eta_{\mu\nu} + h_{\mu\nu} \ , \\ \phi \sim \phi_0 + \delta\phi \ , \\ \lambda \sim \lambda_0 + \delta\lambda \ . \end{cases}$$

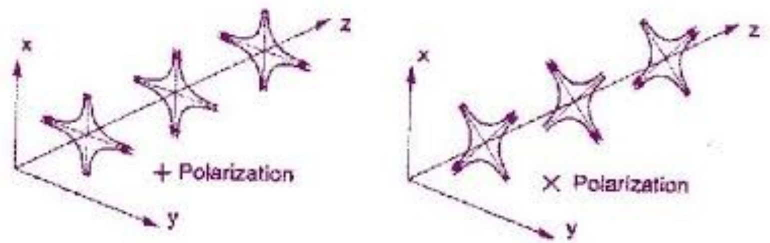
into the field equations, one gets:

$$\begin{aligned} h_{\mu\nu}(x) = & \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k}\, C_{\mu\nu}(\mathbf{k})\, e^{ik_1\cdot x} \\ & + \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \left[\frac{1}{3}\left(\frac{\eta_{\mu\nu}}{2} + \frac{(k_2)_\mu (k_2)_\nu}{k_2^2}\right)\right] \tilde{A}(\mathbf{k})\, e^{ik_2\cdot x} + c.c. \end{aligned}$$

$$h_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k}\, C_{\mu\nu}(\mathbf{k})\, e^{ik_1\cdot x} + \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \left[\frac{1}{3}\left(\frac{\eta_{\mu\nu}}{2} + \frac{(k_2)_\mu (k_2)_\nu}{k_2^2}\right)\right] \tilde{A}(\mathbf{k})\, e^{ik_2\cdot x} + c.c.$$

Gravitational wave in non-local gravity

- $\tilde{A}(\mathbf{k})$ is a square integrable function related to the non-locality
- $(k_2)^\mu = (\omega_2, \mathbf{k})$ is the wave four-vector



The first part is a massless, 2-helicity transverse waves solutions, namely the standard gravitational wave of General Relativity. GR is then recovered when non-local functions vanish

Therefore, for a massless plane wave travelling in +z direction, which propagates at speed c, we have

$$h_{\mu\nu}^{(k_1)}(t,z) = \sqrt{2}\left[\tilde{\epsilon}^{(+)}(\omega_1)\,\epsilon_{\mu\nu}^{(+)} + \tilde{\epsilon}^{(\times)}(\omega_1)\,\epsilon_{\mu\nu}^{(\times)}\right] e^{i\omega_1(t-z)} + c.c.$$

$$\epsilon_{\mu\nu}^{(+)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\mu\nu}^{(\times)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$h_{\mu\nu}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k}\, C_{\mu\nu}(\mathbf{k})\, e^{ik_1\cdot x} + \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \left[\frac{1}{3}\left(\frac{\eta_{\mu\nu}}{2} + \frac{(k_2)_\mu (k_2)_\nu}{k_2^2}\right)\right] \tilde{A}(\mathbf{k})\, e^{ik_2\cdot x} + c.c.$$

Non-locality yields three additional polarizations of the form

$$\epsilon_{\mu\nu}^{(TT)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_{\mu\nu}^{(b)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_{\mu\nu}^{(l)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfying the conditions

$$\mathrm{Tr}\left\{\epsilon^{(a)}\epsilon^{(b)}\right\} = \epsilon_{\mu\nu}^{(a)}\epsilon^{(b)\mu\nu} = \delta^{a,b} \quad \text{with} \quad a, b \in \{+, \times, TT, b, l\}$$

$$\epsilon_{\mu\nu}^{(+)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\mu\nu}^{(\times)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon_{\mu\nu}^{(TT)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_{\mu\nu}^{(b)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_{\mu\nu}^{(l)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Nonetheless, only three (out of five) DOF survive, namely two massless 2-helicity tensor modes and one massive 0-helicity scalar mode, exactly like f(R) gravity

Infinitesimal w.r.t the other modes when the GW speed approach c, namely when the mass of the non-local GW goes to zero

Summing up:

Constraints	Order	Frequency	Polarization	Type	d.o.f.	Modes Petrov Class	Helicity	Mass
$1 + a_1\phi_0 - \lambda_0 \neq 6a_1$	2th	ω_1	2, transverse	tensor	2	$(+), (\times)$	2	0
						N_2		
$1 + a_1\phi_0 - \lambda_0 = 6a_1$	2th	ω_1 ω_2	3, transverse	tensor scalar	3	$(+), (\times), (b)$ N_3	2 0	0 M

Polarizations and modes for gravitational waves in a theory of gravity with non-local corrections.

Main Results provided by GWs in Non-Local Gravity:

- *GWs in Non-Local Gravity exhibit a massive scalar gravitational mode in addition to the standard ones*

$$\epsilon_{\mu\nu}^{(+)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\mu\nu}^{(\times)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_{\mu\nu}^{(b)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- *The model $R + a_1 R\Box^{-1}R$ can be considered as a straightforward extension of General Relativity, where a non-local correction is taken into account*
- *Einstein theory is a particular case occurring when $a_1 = 0$*
- *If we consider deviations of the waves from exactly massless ones propagating at the light speed, two polarization modes are suppressed*

$$\epsilon_{\mu\nu}^{(TT)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon_{\mu\nu}^{(l)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Perspectives:

- Extending the approach to more general terms like $R\Box^{-k}R$

Motivations:

- These models can be ghost-free and their infrared counterparts can be interesting at astrophysical and cosmological scales to address the dark side issues.
- Detecting further modes as the scalar massive one derived here is a major signature to break the degeneracy of modified theories of gravity which could be discriminated at fundamental level

Conclusions

- **NLG** can reproduce, in principle, both UV and IR cosmic evolution

from Noether Symmetries, it is possible:

- *to select physically relevant cosmological models*
- *to derive exact cosmological solutions*
- *To address naturally Dark Energy issues*
- *to constraint solutions by means of experimental observations*

Models can be investigated in the weak-field limit and provide

- *Constraints on S2 star orbit*
- *New potentials to be studied via gravitational lensing*
- *Characteristic lengths could be identified in galaxies and clusters of galaxies*

Gravitational waves in NLG provide a further polarization with respect to the standard ones of GR

- *Finding a new polarization could be a fundamental test for NLG*
- *NLG could contribute to the cosmological stochastic background*
- *A worldwide web of interferometers could contribute to select further polarizations*
- *ET and LISA could be fundamental in identifying these new features*

Perspectives

I. Theoretical perspectives:

- Search for cosmological solutions consistent with cosmic history from UV to IR scales
- Study renormalizability and unitarity of NLG at fundamental level
- Cylindrical BH solutions containing NLG terms
- Quantum cosmology in NLG

II. Observational perspectives:

- Observational constrains of the model free parameters *via* cosmological data, *e.g.* $\mathrm{SNe\ Ia} + \mathrm{BAO} + \mathrm{CC} + H_0$
- Constraining astrophysical scales by S2 star orbit observations by NTT/VLT or EHT
- Refine clusters of galaxies analysis using dust, hot gas, Sunyaev-Zeldovich effect and stellar component
- Possible detection of further gravitational modes