The symmetries of the nonlinear systems

Rodica Cimpoiasu, Radu Constantinescu
University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania

Abstract

The presentation will offer a general overview on what symmetry means in two (apparently) different domains, QFT and nonlinear dynamics. Two fundamental type of symmetries, point-like and gauge-type symmetries will be considered. General approaches using the classical (Lie) or the non-classical (Bluman-Cole) symmetries will be used to investigate some nonlinear mechanical models arising from Quantum contexts.

Keywords: Lie symmetries, invariants, similarity reduction.

1 Aim of the paper

The nonlinear differential equations can accept different classes of solutions and there is no clear algorithm on how to obtain them.

Many approaches and solving methods have been proposed and hundreds of papers claim to report new solutions for various nonlinear equation.

The main question: how the solutions of a given equation can be classified and how we can decide whether a particular solution is really a new one?

Our claim: the independent classes of solutions correspond to the invariant solutions provided by the optimal system of Lie algebras.

2 The integrability of the nonlinear dynamical systems

Sometimes it is difficult to find solutions and it is enough if one can decide on the integrability of the system.

For autonomous Hamiltonian systems, the simplest meaning of integrability consists in the existence of invariant quantities $\{C_i\}$ in involution:

$$\frac{dC_i}{dt} = \{H, C_i\} = 0; \{C_i, C_j\} = 0$$

When the number of these invariants is equal to the number of degrees of freedom of the system, the system is said to be integrable.

Methods that can be used to investigate the integrability of a system:

(i) the inverse scattering method

(ii) the Lax method

(iii) the Painleve approach

(iv) the Hirota's bilinear method

(v) the Lie symmetry method - investigated in this presentation.

3 Symmetries and classes of invariant solutions for NPDEs

3.1 The classical symmetry method

The *classical symmetry method (CSM)*. Let us consider a n-th order partial differential system:

$$\Delta_\nu(x, u^{(n)}[x]) = 0 \tag{1}$$

with the independent variables $x \equiv \{x^i, i = \overline{1,p}\} \subset R^p$, the dependent ones $u \equiv \{u^\alpha, \alpha = \overline{1,q}\} \subset R^q$, and with $u^{(n)}$the set of the partial derivatives of u up to n-th order.

The Lie symmetries represent the set of all the infinitesimal transformations which keep invariant the classes of solutions for the differential system (1). The invariance condition is:

$$X^{(n)}[\Delta] \mid_{\Delta=0} = 0 \tag{2}$$

where the general infinitesimal symmetry operator has the form:

$$X = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{3}$$

The n-th extension is given by:

$$X^{(n)} = X + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{4}$$

with: the multi-indices $J = (j_1, ...j_m)$,with $1 \leq j_m \leq\ p, 1 \leq m \leq\ n$, $u_J^\alpha = \frac{\partial^m u^\alpha}{\partial x^{j_1}\partial x^{j_2}..\partial x^{j_m}}$, and

$$\phi_\alpha^J(x^i, u^{(n)}) = \mathcal{D}_J[\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha] + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha,\ \alpha = \overline{1,q} \tag{5}$$

$$\mathcal{D}_J = \mathcal{D}_{j_1}\mathcal{D}_{j_2}...\mathrm{D}_{j_m} = \frac{d^{\ m}}{dx^{j_1}dx^{j_2}..dx^{j_m}} \tag{6}$$

To the general symmetry generator (3) can be associated characteristic equations of the form:

$$\frac{dx^1}{\xi^1} = ... = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = ... = \frac{du^q}{\phi_q} \tag{7}$$

By integrating the characteristic system of ordinary differential equations (7), the invariants I_r, $r = \overline{1,(p+q-1)}$ of the analyzed system can be found and they become the set of similarity variables is found.

3.2 Generalizations of the classical Lie symmetries

Several generalizations of the classical Lie symmetry method have been proposed:

- the *non-classical symmetry method* (*NSM*) (also referred to as *the conditional method*) of Bluman and Cole,
- the *direct method* of Clarkson and Kruskal,
- the *differential constraint approach* of Olver and Rosenau,
- the *generalized conditional symmetry.*

The basic idea of *NSM* is that (2) should be augmented with the invariance surface condition:

$$Q^\alpha(x, u^{(1)}) \equiv \phi_\alpha(x,u) - \sum_{i=1}^{p}\xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i} = 0,\ \alpha = \overline{1,q} \tag{8}$$

The $q-$tuple $Q = (Q^1, Q^2, ...Q^q)$ is known as the characteristic of the symmetry operator (3).

3.3 The generalized conditional symmetries

The *generalized conditional symmetries (GCS)* or conditional Lie-Bäcklund symmetries applies when the equation (1) can be written as:

$$u_t = E(t, x, u, u_x, ...u_{mx}), \tag{9}$$

The group of Lie symmetries is generated by an evolutionary vector field with η as its characteristic, which admits the canonical form:

$$V = \sum_{k=0}^{\infty} D_x^k \eta \frac{\partial}{\partial u_{kx}} \tag{10}$$

- V is a Lie–Bäcklund symmetry of (9) if and only if $V(u_t - E)\,|_L = 0$,where L is the set of all differential consequences of the equation.

- V is a GCS of (9) if and only if $V(u_t - E)\,|_{L\cap M} = 0$, where M denotes the set of all differential consequences for $D_x^j\eta = 0$,

- If η does not depend on t explicitly, the condition for existing GCS could be expressed in the following terms:

$$\eta' E\,|_{L\cap M} = 0,\ \eta' E = \lim_{\varepsilon\to 0} \frac{d}{d\varepsilon}\eta(u + \varepsilon E), \tag{11}$$

where "prime" denotes here the Fréchet derivative of η along the E direction.

3.4 The inverse symmetry problem (I)

The *direct symmetry approach* investigates the Lie algebra that can be attached to a given equation and allows to obtain the associated invariants and to determine the classes of the invariant solutions that can be accepted by that equation.

The *inverse symmetry problem,* proposed in [18], allows to find the largest class of evolutionary equations which are equivalent from the point of view of their symmetries = more complicated equationss which admit the same Lie symmetry group,

Let us formulate the problem for a $2D$ nonlinear system described by the following general second order partial derivative equation:

$$\begin{aligned} u_t &= A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} + \\ &\quad + E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u) \end{aligned} \tag{12}$$

The general expression of the Lie symmetry operator which leaves (12) invariant can be taken as:

$$X(x,y,t,u) = \frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}. \tag{13}$$

3.5 The inverse symmetry problem (II)

The invariance condition of the equation (12) is given by the relation:

$$\begin{aligned} 0 &= X^{(2)}[u_t - A(x,y,t,u)u_{xy} - B(x,y,t,u)u_x u_y - C(x,y,t,u)u_{2x} - D(x,y,t,u)u_{2y} - \\ &\quad - E(x,y,t,u)u_y - F(x,y,t,u)u_x - G(x,y,t,u)] \end{aligned} \tag{14}$$

The previous relation has the equivalent expression:

$$\begin{aligned} 0 &= -A_t u_{xy} - B_t u_x u_y - C_t u_{2x} - D_t u_{2y} - E_t u_y - F_t u_x - G_t - A_x \xi u_{xy} - B_x \xi u_x u_y - \\ &\quad - C_x \xi u_{2x} - D_x \xi u_{2y} - E_x \xi u_y - F_x \xi u_x - G_x \xi - A_y \eta u_{xy} - B_y \eta u_x u_y - C_y \eta u_{2x} - D_y \eta u_{2y} - \\ &\quad - E_y \eta u_y - F_y \eta u_x - G_y \eta - A_u \phi u_{xy} - B_u \phi u_x u_y - C_u \phi u_{2x} - D_u \phi u_{2y} - E_u \phi u_y - F_u \phi u_x - \\ &\quad - G_u \phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y \end{aligned} \tag{15}$$

The functions $\phi^t, \phi^x, \phi^y, \phi^{2x}, \phi^{2y}, \phi^{xy}$ will be determined using the general formulas from Olver.

3.6 The inverse symmetry problem (III)

By extending the relations (**??**), substituting them into the condition (15) and then equating with zero the coefficient functions of various monomials in derivatives of u, the following system of 11 partial differential equations is obtained:

$$0 = \xi_u;\ 0 = \eta_u;\ 0 = B\eta_x - D\phi_{2u};\ 0 = B\xi_y - C\phi_{2u};$$

$$\begin{aligned}
0 &= A\eta_y - \eta A_y - A_u\phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t \\
0 &= A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u\phi - D_t \\
0 &= -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x\xi - B_u\phi - B_y\eta
\end{aligned}$$

$$\begin{aligned}
0 = &-\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x\xi - E_y\eta - E_u\phi \\
&+A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu}
\end{aligned}$$

$$\begin{aligned}
0 = &-\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x\xi - F_y\eta - F_u\phi \\
&A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu}
\end{aligned}$$

$$\begin{aligned}
0 = &\ \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x\xi - G_y\eta - G_u\phi \\
&-A\phi_{xy} - C\phi_{2x} - D\phi_{2y}
\end{aligned}$$

In the *direct symmetry approach*: one finds the coefficient functions $\xi(x, y, t)$, $\eta(x, y, t)$ and $\phi(x, y, t, u)$ of the Lie operator;

In the *inverse symmetry problem:* one considers $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ as unknown variables.

4 Similarity solutions for the KGF equation

4.1 The KGF Equation

The equation we are going to study in this section is the Klein-Gordon-Fock (KGF) equation with central symmetry:

$$v_{2t} - v_{2r} - \frac{2}{r}v_r + \frac{b}{r^2}v = 0. \tag{16}$$

In the previous equation b is a real parameter.

By changing the dependent variable, $v(r, t) = u(r, t)/r$, the following reduced form of Eq. (16) is obtained:

$$u_{2t} - u_{2r} + \frac{b}{r^2}u = 0. \tag{17}$$

Eq. (17) belongs to the class of wave equations with time-independent potential:

$$u_{2t} - u_{2r} + V(r)u = 0. \tag{18}$$

where $u(t, r) \in C^2(R^2, R^1)$ and the potential $V(r) \in C^2(R^1, R^1)$.

In the case $b = 0$, Eq. (17) becomes the d'Alembert equation.

4.2 The GCS method for the KGF Equation

The operator which generates the GCS group for (17) takes the form:

$$V = \eta\frac{\partial}{\partial u} + (D_r\eta)\frac{\partial}{\partial u_r} + (D_t\eta)\frac{\partial}{\partial u_t} + (D_{2r}\eta)\frac{\partial}{\partial u_{2r}} + (D_{2t}\eta)\frac{\partial}{\partial u_{2t}} + ... \tag{19}$$

The invariance condition is:

$$\frac{b}{r^2}\eta - D_{2r}\eta + D_{2t}\eta\,|_{L\cap M} = 0. \tag{20}$$

The Eq. (17) admits GCSs if and only if

$$D_{2t}\eta = 0, \tag{21}$$

The characteristic can be taken as:

$$\eta[r,u] = u_{2r} - H(u)u_r^2 - P(r,u)u_r - R(r,u). \tag{22}$$

Taking into account the surface condition $\eta = 0$, we obtain $H = 0$, $P(r,u) \equiv P(r)$, $R(r,u) = Q(r)u + M(r)$.

For $P(r) = \frac{k}{r}$, $Q(r) = \frac{m}{r^2}$.with k and m arbitrary constants, the remaining function $M(r)$ must verify the ordinary differential equation:

$$-\frac{2k}{r^2}M + M" - \frac{b}{r^2}M = 0. \tag{23}$$

4.3 Invariant classes of KGF solutions

An exhaustive study on the invariant classes of KGF solutions was presented in [31]. We remind here only one class obtained for the values of the parameters $b \neq 0$, $b \neq m$ and $4b = (k-3)(k-1)$. In this case:

$$P(r) = \frac{k}{r},\ Q(r) = -\frac{(k-1)(k+3)}{4r^2},\ M(r) = c_1 r^{-\frac{1+k}{2}} + c_2 r^{\frac{k+3}{2}}. \tag{24}$$

with c_1, c_2 arbitrary parameters, and $m = -\frac{(k-1)(k+3)}{4}$.

The GCS operator takes the form:

$$V_I = \left[u_{2r} - \frac{k}{r}u_r + \frac{(k-1)(k+3)}{4r^2}u - c_1 r^{-\frac{1+k}{2}} - c_2 r^{\frac{k+3}{2}}\right]\frac{\partial}{\partial u}. \tag{25}$$

By solving the invariance surface condition $\eta = 0$, we come to the solution of the KGF equation as:

$$\begin{aligned} u(t,r) \quad = \quad & f(t)r^{(k-1)/2} + g(t)r^{(k+3)/2} + r^{(k+7)/2} + \frac{8c_1}{k(k-2)}r^{-(k-3)/2} - \\ & -\frac{2[k(k+5)(k+7) - 2c_1(k-3)(k-1)]}{k(k+1)(k+3)}r^{(k+3)/2} + \\ & \frac{(k-2)(k+5)(k+7) - 4c_1(k-3)(k-1)}{(k-3)(k-1)(k-2)}r^{(k-1)/2}. \end{aligned} \tag{26}$$

The functions $f(t)$ and $g(t)$ must be polynomials of second order in t.

5 The Lie symmetry problems for 2D reaction-diffusion equation

5.1 The reaction-diffusion equation

The reaction-diffusion equation is a second order parabolic equation which describes physical phenomena due to the processes of reaction, diffusion and convection.

In the simpler case when the diffusion coefficient is variable, the convection velocity is constant and there are no sources or sinks, the equation takes the form:

$$u_t = uu_{2x} + uu_{2y} - vu_x \tag{27}$$

with diffusion coefficient u and convective velocity $v = const.$ along to the Ox direction.

It is easy to remark that (27) results from the general class of equations (12) by choosing the particular functions:

$$\begin{aligned} C(x,y,t,u) &= D(x,y,t,u) = u,\ F(x,y,t,u) = -v \\ A(x,y,t,u) &= B(x,y,t,u) = E(x,y,t,u) = G(x,y,t,u) \equiv 0 \end{aligned} \tag{28}$$

5.2 Lie symmetries for the 2D reaction-diffusion equation

The general determining system for the symmetry generators has in this case the solution:

$$\xi = \frac{c_1}{2}(x - vt) + c_2 y + c_3,\ \eta = \frac{c_1}{2}y - c_2(x - vt) + c_4,\ \phi = c_1 u \tag{29}$$

The Lie symmetry generator takes the expression:

$$X(x,y,t,u) = \frac{\partial}{\partial t} + \left(\frac{c_1}{2}(x - vt) + c_2 y + c_3\right)\frac{\partial}{\partial x} + \left(\frac{c_1}{2}y - c_2(x - vt) + c_4\right)\frac{\partial}{\partial y} + c_1 u\frac{\partial}{\partial u} \tag{30}$$

Consequently, the nonlinear reaction-diffusion equation (27) admits the $4-$dimensional Lie algebra spanned by the operators shown below:

$$\begin{aligned} X_1 &= \left(\frac{x - vt}{2}\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \\ X_2 &= y\frac{\partial}{\partial x} - (x - vt)\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial x},\ X_4 = \frac{\partial}{\partial y} \end{aligned} \tag{31}$$

When the Lie algebra of these operators is computed, the only non-vanishing relations are:

$$[X_3, X_1] = \frac{1}{2}X_3, [X_4, X_1] = X_4,\ [X_2, X_3] = X_4,\ [X_4, X_2] = X_3 \tag{32}$$

5.3 Invariant solutions for the reaction-diffusion equation

Each operator from (31) can generate invariant solutions of the model. The invariant classes of solutions correspond to the set of optimal subalgebras that has in this case the dimension 3. Such an optimal subalgebra is given by $\{X_2, X_3, X_1 + \alpha X_2 + \beta X_4\}$.

For the operator X_2, the characteristic equations generate 3 invariants, $I_1 = t$, $I_2 = vtx - \frac{x^2}{2} - \frac{y^2}{2}$, $I_3 = u$, and the similarity solution:

$$u(t,x,y) = \frac{2vtx - x^2 - y^2 - v^2t^2 + 2q_1}{4t + 2q_2} \tag{33}$$

The operator X_3 yields also 3 invariants, $I_1 = t$, $I_2 = y$, $I_3 = u$, and the similarity solution:

$$u(t,y) = \frac{\frac{q_1}{2}y^2 + q_3 y + q_4}{q_2 - q_1 t}. \tag{34}$$

The third operator of the considered optimal subalgebra, $X_1 + \alpha X_2 + \beta X_4$, generates other 3 invariants and the invariant solution:

$$u(t,x,y) = -\frac{1}{2\left(\alpha^2 + \frac{1}{4}\right)t - \gamma}\left[\frac{y^2}{2} + \alpha(vt - x) + \beta\right]^2 \tag{35}$$

5.4 Inverse symmetry problem for the $2D$ reaction-diffusion equation

Our aim is now to find the class of equations with generic form (12) which admits the same symmetries with those corresponding to $2D$ nonlinear convective-diffusion equation (27). Consequently, we have to impose that the coefficient functions (29) which determine the base of symmetry operators (31) verify the general determining system (**??**).

The solutions of differential system (**??**) describe the coefficient functions of the general evolutionary equation (12) as follows:

$$\begin{aligned} A &= B = 0,\ C = D = c_3 u, \\ E(u) &= \sqrt{u}\left[c_4 \cos\left(\frac{c_2}{c_1}\ln(u)\right) - c_5 \sin\left(\frac{c_2}{c_1}\ln(u)\right)\right] \\ F(u) &= \sqrt{u}\left[c_4 \sin\left(\frac{c_2}{c_1}\ln(u)\right) + c_5 \cos\left(\frac{c_2}{c_1}\ln(u)\right) - v\right] \\ G(u) &= c_6 u \end{aligned} \tag{36}$$

where c_j, $j = \overline{1,6}$ and v are arbitrary constants.

In particular, for $c_3 = 1$, $c_4 = c_5 = c_6 = 0$ and arbitrary c_1 and c_2, the solution (36) generates the $2D$ nonlinear reaction-diffusion equation (27) discussed above.

6 Conclusions

- The direct symmetry method allows to obtain the classes of invariant solutions for a nonlinear differential equation using the optimal set of Lie subalgebras. Steps to be followed suppose to determine:

(i) the general Lie algebra;
(ii) the optimal sets of independent generators (subalgebras);
(iii) the invariant solutions corresponding to each set.

- The inverse symmetry method allows to find the largest class of nonlinear differential equations which belong to the same class as a given equation in the sense of the symmetries they observe.

- Both direct and inverse methods were applied to two important examples of nonlinear $2D$ partial derivative equations:

- The *KGF equation*, with an optimal system of subalgebras of dimension 4, same as the whole symmetry algebra.
- The *reaction-diffusion equation* with a maximal optimal subalgebra of dimension 3, despite the existence of 4 independent symmetry operators.

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