# On the example of the geometrically integrable map with ramified chaotic attractor

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# Main definitions

Let a discrete dynamical system with the phase space $M$ be given by a map $F : M \to M$. Let $B \subseteq M$ be *an absorbing domain* in $M$, i.e., the inclusion $F(\overline{B}) \subset B$ holds, where $\overline{(\cdot)}$ is the closure of a set.

*The maximal attractor $A_{max}$* in the absorbing domain $B$ is said to be the set

$$A_{max} = \bigcap_{n=1}^{+\infty} F^n(B).$$

An invariant set $A$ is said to be *an attractor* of $F$ if there exists an absorbing domain, for which $A$ is *the maximal attractor*.

A connected compact Hausdorff space $A$ is called *a continuum*.
Let $A$ be a continuum, $z$ be a point in $A$. We say that $z$ is a point of an order $n \in \mathbb{N}$ if $z$ is a unique common endpoint of every two of exactly $n$ arcs contained in $A$.
By *a ramified continuum* we mean a continuum admitting points of an order $n \geqslant 3$. Such points are called *ramification points*.

# Maps under consideration, I

Let a self-map $F$ of the phase space $M$ (equipped with $x$ and $y$ coordinates) be presented in the form

$$F(x, y) = (f(x) + \mu(x, y),\ g_x(y)), \text{ where } g_x(y) = g(x, y). \quad (1)$$

If $M$ is the plane, then we obtain Hénon map for

$$f(x) + \mu(x, y) = 1 - ax^2 + y,\ \ g_x(y) = by;$$

and Lozi map for

$$f(x) + \mu(x, y) = 1 - a|x| + y,\ \ g_x(y) = by.$$

If $M$ is a cylinder (or a torus), then we obtain Belykh map for

$$f(x) + \mu(x, y) = x + h(x) + y,\ \ g_x(y) = \lambda(h(x) + y).$$

# Maps under consideration, II

Let $M$ be a compact two-dimensional cylinder $M = S^1 \times I_2$. Here $S^1$ is a circle, $I_2$ is a compact interval of the real line. We consider $C^1$-smooth maps (1) close (in the $C^1$-norm) to skew products on the cylinder $M$ and so that a function $\mu$ satisfies zero boundary conditions.

A map $\varphi \in C^1(S^1)$ is said to be $\Omega$-*stable* (in the $C^1$-norm) if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for a map $\psi \in B^1_{1,\varepsilon}(\varphi)$ one can find $\delta$-close in the $C^0$-norm to the identity map homeomorphism $h : \Omega(\varphi) \to \Omega(\psi)$ satisfying the equality

$$h \circ \varphi_{|\Omega(\varphi)} = \psi_{|\Omega(\psi)} \circ h. \tag{2}$$

If equality (2) is fulfilled not only for maps $\varphi$ and $\psi$ on the their nonwandering sets $\Omega(\varphi)$ and $\Omega(\psi)$ respectively, but on the circle $S^1$, than the map $\varphi$ is said to be $C^1$-*structurally stable*.

# Some definitions of one-dimensional dynamics

Let $S^1 = \{x \in \mathbb{C} : |x| = 1\} = \{e^{2\pi it} : t \in \mathbb{R}^1\}$. Then for every continuous map $\varphi : S^1 \to S^1$ there is a continuous map $\widehat{\varphi} : \mathbb{R}^1 \to \mathbb{R}^1$ satisfying

$$\varphi \circ exp = exp \circ \widehat{\varphi},$$

where $exp\, t = e^{2\pi it}$ for every $t \in \mathbb{R}^1$.
The above continuous map $\widehat{\varphi}$ is said to be *a lifting* of the circle map $\varphi$.

*The degree* of a map $\varphi$ is said to be an integer number

$$deg\, \varphi = \frac{1}{n}(\widehat{\varphi}(t+n) - \widehat{\varphi}(t)),$$

that does not depend on $t \in \mathbb{R}^1$; moreover, it is defined for an integer number $n$ for any lifting $\widehat{\varphi} : \mathbb{R}^1 \to \mathbb{R}^1$ of the circle map $\varphi$.

# The main result of one-dimensional dynamics

**Theorem.** *In the space $C^1(S^1)$ there is an open everywhere dense set $\mathbf{L}$ of maps that equals the union of two subsets $\mathbf{L_1}$ and $\mathbf{L_2}$, where $\mathbf{L_1}$ is the set of $\Omega$-stable maps with a completely disconnected nonwandering set, and $\mathbf{L_2}$ is the set of structurally stable maps such that every $\varphi \in \mathbf{L_2}$ is topologically conjugate with an expanding map of the same degree $deg\, \varphi$, where $|deg\, \varphi| > 1$.*

We interpret a circle $S^1$ as the unit interval $[0, 1]^*$ with identified points 0 and 1. Let $I_2 = [a, b]$.
We consider the class $C^1_L(M)$ of $C^1$-smooth maps (1), where $f \in \mathbf{L}$, and $\mu$ satisfies the condition: $(i_\mu)$ $\mu(0, y) = \mu(1, y) = \mu(x, a) =$ $= \mu(x, b) = 0$ for $x \in [0, 1]^*$, $y \in [a, b]$. Then $T^1_L(M) \subset C^1_L(M)$, where $T^1_L(M)$ is the space of $C^1$-smooth skew products ($\mu(x, y) \equiv 0$ in (1)) with quotient from the set $L$.

Let the base of the $C^1$-topology in $C^1_L(M)$ be given by the family of $\varepsilon$-balls $B^1_\varepsilon(F)$ for every $F \in C^1_L(M)$ and $\varepsilon > 0$.

# The condition of smallness of the function $\mu$

We also need the class $C^1(M,\, S^1)$ of $C^1$-smooth maps of the cylinder $M$ into the circle $S^1$ endowed with the standard $C^1$-norm $||\cdot||_{C^1(M,\, S^1)}$. Then for every $y \in I_2$ we have:

$$||\mu||_{C^1(S^1)} \leqslant ||\mu||_{C^1(M,\, S^1)}.$$

Let $\delta > 0$. Since $f \in \mathbf{L}$, then we find $\varepsilon > 0$, using the definition of the $\Omega$-stability (or the structural stability). The "condition of smallness"in the $C^1$-norm means that the following inequality is valid:

$(ii_{\mu,\,\varepsilon})$ $||\mu||_{C^1(M,\, S^1)} < \varepsilon.$

# Definition of $C^r$-local one-dimensional lamination ($r \geqslant 0$)

Let $L_\alpha$ be a $C^r$-curve, $L_\alpha \subset M$, $L_\alpha$ be $C^r$-regularly embedded to $M$.

Let $A$ be a subset of $M$ satisfying $A = \bigcup\limits_\alpha L_\alpha$. Here $\alpha$ belongs to an index set; $C^r$-curves $\{L_\alpha\}_\alpha$ are pairwise disjoint. The family of curves $\{L_\alpha\}_\alpha$ is said to be *one-dimensional $C^r$- local lamination (without singularities)*, if for every point $z \in A$, $z = z(x, y)$, there exist a neighborhood $U(z) \subset M$ and a $C^r$-diffeomorphism for $r \geqslant 1$ or homeomorphism for $r = 0$ $\chi : U(z) \to \mathbb{R}^2$ (here $\mathbb{R}^2$ is the plane) such that every connected component of the intersection $U(z) \bigcap L_\alpha$ (if it is not empty) is mapping by means of $\chi$ into a straight line such that

$$\chi_{|U(z)\bigcap L_\alpha} : U(z) \bigcap L_\alpha \to \chi(U(z) \bigcap L_\alpha)$$

is a $C^r$-diffeomorphism for $r \geqslant 1$ or homeomorphism for $r = 0$ on the image.

# Existence of a $C^1$-smooth local lamination

**Theorem 1.** *Let* $\Phi \in T^1_L(M)$ *be a map of the form*

$$\Phi(x, y) = (f(x), g_x(y)). \tag{3}$$

*Let* $\delta > 0$. *Then there is an* $\varepsilon$*-neighborhood* $B^1_\varepsilon(\Phi)$ *of the map* $\Phi$ *in the space* $C^1_L(M)$ *such that every map* $F \in B^1_\varepsilon(\Phi)$ *obtained from* $\Phi$ *by means of the* $C^1$*-smooth perturbation* $\mu = \mu(x, y)$, *where* $\mu$ *satisfies the condition* $(ii_{\mu,\varepsilon})$, *has an invariant* $C^1$*-smooth local lamination* $L^1_{loc}(F)$, *which is a lamination for* $f \in \mathbf{L_1}$, *and a foliation for* $f \in \mathbf{L_2}$. *Fibres of this local lamination start from the points of the set* $\Omega(f) \times \{a_2\}$ *and are pairwise disjoint graphs of* $C^1$*-smooth functions* $x = x(y)$ *on the interval* $I_2$. *Moreover, every curvilinear fibre is* $\varepsilon'$*-close in the* $C^1$*-norm (for some* $\varepsilon' > 0$, $\varepsilon' = \varepsilon'(\delta)$*) to the vertical closed interval that starts from the same initial point of the set* $\Omega(f) \times \{a_2\}$ *just as the curvilinear fibre.*

# The definition of geometric integrability

**Definition 1**. *A map $F : M \to M$ is said to be geometrically integrable on a nonempty $F$-invariant set $A(F) \subseteq M$ if there exist a self-map $\psi$ of an arc $J \subseteq M_1$ and $\psi$-invariant set $B(\psi) \subseteq J$ such that the restriction $F_{|A(F)}$ is semiconjugate with the restriction $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$, i.e., the following equality holds:*

$$H \circ F_{|A(F)} = \psi_{|B(\psi)} \circ H.$$

*The map $\psi_{|B(\psi)}$ is said to be the quotient of $F_{|A(F)}$.*

**Remark 1**. In the framework of our approach the concept of geometric integrability is introduced for some multifunctions (see L.S. Efremova, The Trace Map and Integrability of the Multifunctions, J. Phys.: Conf. Ser., **990** (2018), 012003).

# The geometric criterion of integrability on an invariant set

We use further first $pr_1$ and second $pr_2$ natural projections.
**Theorem 2**. Let $F$ be a self-map of $M$, $A(F)$ be a closed $F$-invariant subset of $M$ satisfying

$$pr_2(A(F)) = I_2. \tag{4}$$

Let $J \subseteq S^1$ be an arc, $\psi$ be a self-map of $J$, $B(\psi)$ be a closed $\psi$-invariant subset of $J$.
Then $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in M_2$ the map $H$ is an injection on $x$, if and only if $A(F)$ is the support of a continuous invariant lamination for $A(F) \neq M$ (of a continuous invariant foliation for $A(F) = M$) with fibres $\{\gamma_{x'}\}_{x' \in B(\psi)}$ that are pairwise disjoint graphs of continuous functions $x = x_{x'}(y)$ for every $y \in I_2$. Moreover, the inclusion $F(\gamma_{x'}) \subseteq \gamma_{\psi(x')}$ holds.

# On the geometric integrability of maps under consideration

Define a curvilinear projection $H$ on the support of the local lamination $L^1_{loc}(F)$. Let $(x, y)$ be a point of the support of $L^1_{loc}(F)$. Then there is a fibre $\gamma_{x'}$ such that $(x, y) \in \gamma_{x'}$, where $x' \in \Omega(f)$. We set

$$H(x, y) = x'.$$

**Theorem 3.** Let $\Phi \in T^1_L(M)$ be a map of the form (3). Let $\delta > 0$. Then there is an $\varepsilon$-neighborhood $B^1_\varepsilon(\Phi)$ of the map $\Phi$ in the space $C^1_L(M)$ such that every map $F \in B^1_\varepsilon(\Phi)$ obtained from $\Phi$ by means of the $C^1$-smooth perturbation $\mu = \mu(x, y)$, where $\mu$ satisfies the condition $(ii_{\mu, \varepsilon})$, is geometrically integrable on the support of the local lamination $L^1_{loc}(F)$ with the quotient $f_{|\Omega(f)}$ by means of a $C^1$-smooth curvilinear projection $H$ such that for every $y \in I_2$ the map $H$ is an injection on $x$ satisfying the inequality

$$\frac{\partial}{\partial x} H(x, y) \neq 0$$

.

# The analytic criterion of integrability on an invariant set

**Theorem 4.** *Let $F : M \to M$, $A(F)$ be a nonempty closed $F$-invariant subset of $M$ satisfying the (4). Let $J \subseteq S^1$ be an arc, $\psi$ be a self-map of $J$, $B(\psi)$ be a closed $\psi$-invariant subset of $J$. Then $F_{|A(F)}$ is the geometrically integrable map with the quotient $\psi_{|B(\psi)}$ by means of a continuous surjection $H : A(F) \to B(\psi)$ such that for every $y \in I_2$ the map $H$ is an injection on $x$, if and only if there is a homeomorphism $\widetilde{H}$ that maps the set $A(F)$ on the set $B(\psi) \times I_2$ and reduces the restriction $F_{|A(F)}$ to the skew product $\Psi_{|B(\psi)\times I_2}$ satisfying the equality*

$$\Psi_{|B(\psi)\times I_2}(u, v) = (\psi_{|B(\psi)}(u), g_{x'}(v)), \;\; x' = pr_1 \circ \widetilde{H}^{-1}(u, v).$$

*Here $\widetilde{H}^{-1} : B(\psi) \times I_2 \to A(F)$ is the inverse homeomorphism for $\widetilde{H}$,*

$$\textit{and} \;\; \widetilde{H}(x, y) = (H(x, y), y), \; \textit{for all} \; (x, y) \in A(F).$$

# $C^1$-smooth conjugacy of maps (1) to skew products

**Theorem 5.** *Let $\Phi \in T^1_L(M)$ be a map of the form (3). Let $\delta > 0$. Then there is an $\varepsilon$-neighborhood $B^1_\varepsilon(\Phi)$ of the map $\Phi$ in the space $C^1_L(M)$ such that every map $F \in B^1_\varepsilon(\Phi)$ obtained from $\Phi$ by means of the $C^1$-smooth perturbation $\mu = \mu(x, y)$, where $\mu$ satisfies the condition $(ii_{\mu,\varepsilon})$, possesses the following property: the restriction $F_{|L^1_{loc}(F)}$ is $C^1$-smoothly conjugate under the $C^1$-diffeomorphism $\widetilde{H} : L^1_{loc}(F) \to \Omega(f) \times I_2$ to the skew product $\Psi_{|\Omega(f)\times I_2}$ satisfying:*

$$\Psi_{|\Omega(f)\times I_2}(u, v) = (f_{|\Omega(f)}(u), g_{x'}(v)), \;\; x' = pr_1 \circ \widetilde{H}^{-1}(u, v).$$

*Here $\widetilde{H}^{-1} : \Omega(f) \times I_2 \to L^1_{loc}(F)$ is the inverse $C^1$-diffeomorphism for $\widetilde{H}$,*

$$\text{and} \;\; \widetilde{H}(x, y) = (H(x, y), y), \; \text{for all} \; (x, y) \in L^1_{loc}(F).$$

# Example of maps with one-dimensional ramified attractors, I

Let $C^1$-smooth maps $F_k : M \to M$ ($k > 1$, $k \in \mathbf{N}$, $I_2 = [0, 1]$) be so that

$$F_k(x, y) = (f_k(x) + \mu(x, y), g_x(y)), \quad f_k(x) = kx \,(mod\, 1).$$

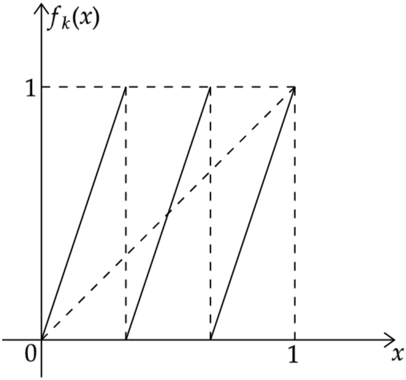



Рис.: The graph of $f_k$ for $k = 3$.

# Example of maps with one-dimensional ramified attractors, II

To construct maps $g_x(y)$ on $S^1 \times [0,\,1]$ ($S^1 = [0,\,1]^*$) for the cylinder map $F_k$, we use a $C^1$-smooth Urysohn function $y = h_k(x)$, where $x \in [0,\,1]^*$ such that

$$h_k(0) = h_k(1) = \tfrac{3}{4},\ \ h_k(\tfrac{1}{k}\left[\tfrac{k}{2}\right]) = 0;$$
$$h_k'(0) = h_k'(1) = h_k'(\tfrac{1}{k}\left[\tfrac{k}{2}\right]) = 0.$$

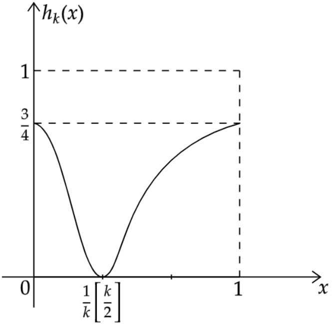



Рис.: The graph of Urysohn function $h_k$ for $k = 3$.

## Example of maps with one-dimensional attractors, III

Then for small enough $\delta$ the graph of the function $y = h_k(x)$ intersects every fibre $\gamma_{x'}$ in the unique point without tangency, and the equality $y = h_k(x_{x'}(y))$ is equivalent to $y = y(x')$, where $x' \in S^1$; moreover, the function $y = y(x')$ is $C^1$-smooth on $S^1$. We use two connected sets

$$D' = \{(x,\, y) \in \bigcup_{x' \in [0,1]^*} \gamma_{x'} \,:\, 0 \leqslant y \leqslant y(x')\};$$
$$D'' = \{(x,\, y) \in \bigcup_{x' \in [0,1]^*} \gamma_{x'} : y(x') < y \leqslant 1\}.$$

We set

$$g_x(y) = \begin{cases} y, & \text{if} \quad (x,\, y) \in \gamma_{x'} \bigcap D', \\ h_k(x) + \sin(y - h_k(x)), & \text{if} \quad (x,\, y) \in \gamma_{x'} \bigcap D''. \end{cases}$$

# Example of maps with one-dimensional attractors, IV

By Theorem 5 maps $F_k$ $(k > 1)$ $C^1$-smoothly conjugate to skew products $\widehat{\Phi}_k = (f_k, \widehat{g}_{x'})$ for all $x' \in S^1$.
Graphs of fibre maps for different $x' \neq 1/k[k/2]$ are presented in the picture.

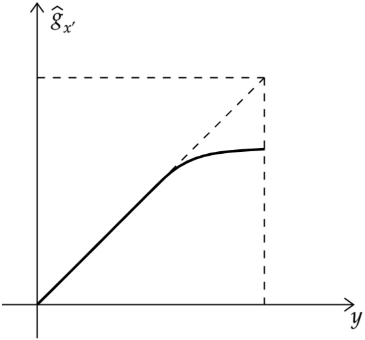



Рис.: The graph of $\widehat{g}_{x'}$ for $x' \neq 1/k\left[k/2\right]$.

# Description of the nonwandering set of $\widehat{\Phi}_k$: $\Omega$-function

We use the topological space $2^{I_2}$ of all closed subsets of the segment $I_2$ endowed with the exponential topology, i. e. the weakest topology in which the sets $2^A$ are open in $2^{I_2}$ for open sets $A$, and closed in $2^{I_2}$ for closed sets $A$. Here $2^A$ means the set of all closed subsets lying in $A \subset I_2$.

**Definition 6.** *The $\Omega$-function* of a skew product $\Phi \in T^0(M)$ is defined to be the function $\Omega^\Phi : \Omega(\varphi) \to 2^{I_2}$ such that for any $x \in \Omega(\varphi)$ the equality holds:

$$\Omega^\Phi(x) = \big(\Omega(\Phi)\big)(x),$$

where $(\Omega(\Phi))(x)$ is the slice of the nonwandering set $\Omega(\Phi)$ by the fibre over $x$.

*The slice* $(A')(x)$ of a set $A' \subseteq M$ by the vertical fibre over a point $x \in S^1$ is the following set:

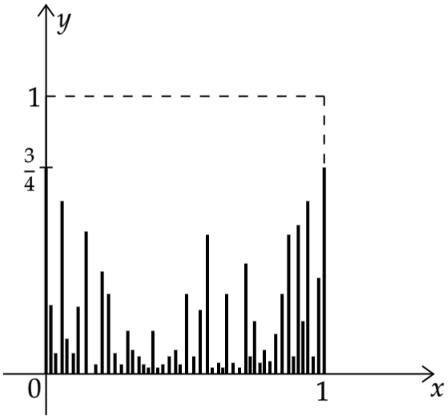
$$\big(A'\big)(x) = \{y \in I_2 : (x, y) \in A'\}.$$

# Properties of $\Omega(\widehat{\Phi}_k)$ in terms of the $\Omega$-function, I

**Theorem 7.** *The $\Omega$-function $\Omega^{\widehat{\Phi}_k} : [0,\, 1]^* \to 2^{[0,\,1]}$ of a skew product $\widehat{\Phi}_k \in T^1_L(M)$ $(k \geqslant 2)$ possesses the following properties:*
(7.1) *it is upper semicontinuous;*
(7.2) *in each point $x' \in [0,\, 1]^*$ with $f_k$-trajectory, the closure of which does not contain $x'_* = 1/k\left[k/2\right]$, $\Omega^{\widehat{\Phi}_k}$ is discontinuous function, and $\Omega^{\widehat{\Phi}_k}(x') = [0,\, (c^k)(x')]$, where $c^k(x') > 0$;*
(7.3) *in each point $x' \in [0,\, 1]^*$ with $f_k$-trajectory, the closure of which contains $x'_* = 1/k\left[k/2\right]$, $\Omega^{\widehat{\Phi}_k}$ is continuous function, and $\Omega^{\widehat{\Phi}_k}(x') = \{0\}$;*
(7.4) *the graph of the $\Omega$-function $\Omega^{\widehat{\Phi}_k}$ in $M$ is one-dimensional topological continuum; its ramification points form an everywhere dense subset of $[0,\, 1]^* \times \{0\}$ of the cardinality equal to the continuum $c$, and each ramification point has the order 3 (see the following Figure);*

# Properties of $\Omega(\widehat{\Phi}_k)$ in terms of the $\Omega$-function, II

(7.5) the graph of the $\Omega$-function $\Omega^{\widehat{\Phi}_k}$ in $M$ is the global chaotic attractor of $\widehat{\Phi}_k$ that coincides with the closure of the set $Per(\widehat{\Phi}_k)$ and with the closed set $\bigcup\limits_{(x',y)\in M} \omega_{\widehat{\Phi}_k}((x', y))$; moreover, the second eigenvalue of each $\widehat{\Phi}_k$-periodic point equals 1.



# The concept of dense intermittency of attraction sets

**Definition 8.** We say that a map $F : M \to M$ with a global chaotic attractor $A$ possesses the property of *dense intermittency* (in the complement to the attractor) of attraction sets of different $\omega$-limit sets, the union of which coincides with the global attractor $A$, if every point $(\overline{x}', \overline{y}) \in M \setminus A$ has a neighborhood $U((\overline{x}', \overline{y})) \subset M \setminus A$ satisfying the property:
in the intersection of $U((\overline{x}', \overline{y}))$ with an arbitrary circle $y = const$ (if it is not empty) between arbitrary two different points $(x_1'; y)$ and $(x_2'; y)$ of attraction sets of different $\omega$-limit sets there is a point $(x_3'; y)$ that belongs to the attraction set of an $\omega$-limit set different from each of previous two $\omega$-limit sets.

# Theorem about the global ramified attractor, I

**Theorem 9.** *Let $\Phi_k \in T^1_L(M)$ be a map of the form (3) with the quotient $f_k$ and fibres maps described above. Let $\delta$ be a positive number, $\delta < \overline{\delta}$ (for some $\overline{\delta} > 0$). Then there is an $\varepsilon$-neighborhood $B^1_\varepsilon(\Phi_k)$ of the map $\Phi_k$ in the space $C^1_L(M)$ such that every map $F_k \in B^1_\varepsilon(\Phi_k)$ obtained from $\Phi_k$ by means of the $C^1$-smooth perturbation $\mu = \mu(x, y)$, where $\mu$ satisfies the condition $(ii_{\mu,\varepsilon})$, has one-dimensional global chaotic attractor $A(F_k)$, which is a ramified continuum with everywhere dense ramification points set of the cardinality c on the invariant circle $S^1 \times \{0\}$. Moreover, the following properties are fulfilled:*

(9.1) $A(F_k) = \overline{Per(F_k)} = \bigcup_{(x,y)\in M} \omega_{F_k}(x, y)$, *and all $F_k$-periodic points are not hyperbolic;*

# Theorem about the global ramified attractor, II

(9.2) *$A(F_k)$ consists of two types of $C^1$-smooth arcs: on the circle $S^1 \times \{0\}$ the map $F_k$ is mixing, and on each nondegenerate arc of the second type (the set of these arcs has continuum cardinality) the map $F_k^n$ is not mixing for every $n \geqslant 1$;*

(9.3) *$F_k$ possesses the property of dense intermittency (in the complement to the attractor) of attraction sets of different $\omega$-limit sets, the union of which coincides with the global attractor $A(F_k)$.*

## References


L.S. Efremova, Ramified continua as global attractors of $C^1$-smooth self-maps of a cylinder close to skew products, J. Difference Eq. and Appliq., **28** (2023) DOI: https://doi.org/10.1080/10236198.2023.2204144