Integrable Models of the Chaplygin ball

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Chaplygin ball

One of the most famous solvable problems in nonholonomic mechanics describes rolling without slipping of a balanced, dynamically nonsymmetric ball over a horizontal plane.

Let O_{B}, a, m, $\mathbb{I} = \mathrm{diag}(A, B, C)$, be the center, radius, mass and the inertia operator of a ball B. The equations of motion in the frame attached to the ball can be written in the form

$$\dot{\vec{\mathrm{k}}} = \vec{\mathrm{k}} \times \vec{\Omega}, \qquad \dot{\vec{\Gamma}} = \vec{\Gamma} \times \vec{\Omega},$$

where Ω is the angular velocity of the ball,
$\vec{\mathrm{k}} = \mathbb{I}\vec{\Omega} + D\vec{\Omega} - D\langle\vec{\Omega}, \vec{\Gamma}\rangle\vec{\Gamma}$ is the angular momentum of the ball with respect to the point of contact, and $D = ma^2$.

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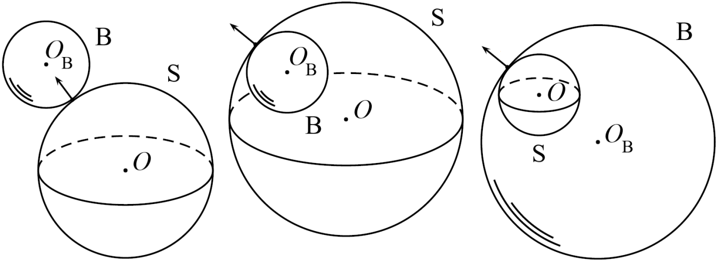
Rolling of the ball B with center O_{B} over the sphere S with center O: three scenarios



(i) rolling of B over the outer surface of S and S is outside B (see the leftmost part of Fig);

(ii) rolling of B over the inner surface of S ($b > a$)(see the central part of Fig);

(iii) rolling of B over the outer surface of S and S is within B; in this case $b < a$ and the rolling ball B is a spherical shell (see the rightmost part of Fig).

Let $\varepsilon = b/(b \pm a)$, where we take "$+$"for the case (i) and "$-$"in the cases (ii) and (iii). The equations of motion:

$$\dot{\vec{\mathrm{k}}} = \vec{\mathrm{k}} \times \vec{\Omega}, \qquad \dot{\vec{\gamma}} = \varepsilon \vec{\Gamma} \times \vec{\Omega}.$$

When b tends to infinity, then ε tends to 1 and $\vec{\Gamma}$ tends to the unit vector that is constant in the fixed reference frame. This way, for $\varepsilon = 1$, we obtain the equations of motion of the Chaplygin ball rolling over the plane orthogonal to $\vec{\Gamma}$.
Remarkably, for $\varepsilon = -1$, which is the case (iii) above with $a = 2b$, the problem is integrable.

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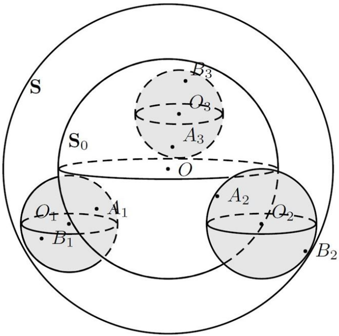
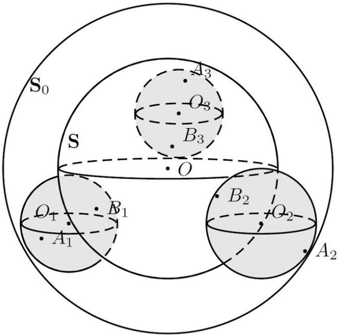
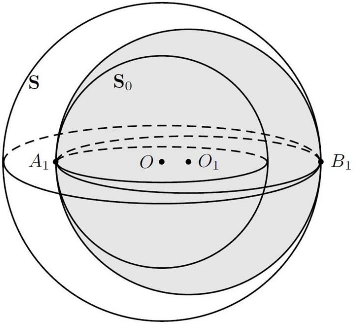
Spherical ball bearing in four configuration. Cases I and II

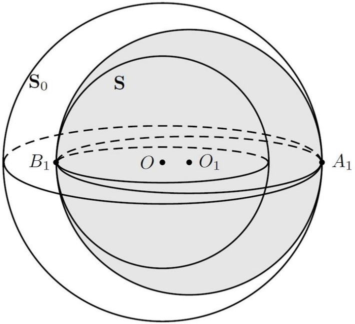
We consider n homogeneous balls $\mathrm{B}_1, \ldots, \mathrm{B}_n$ with centers $O_1, \ldots, O_n$ and the same radius r roll without slipping around a fixed sphere S_0 with center O and radius R. A dynamically nonsymmetric sphere S of radius $\rho = R \pm 2r$ with the center that coincides with the center O of the fixed sphere S_0 rolls without slipping over the moving balls $\mathrm{B}_1, \ldots, \mathrm{B}_n$.





Spherical ball bearing, case III and case IV ($\rho = 2r - R$)





V. Dragović, B. Gajić, B.J, *Spherical and Planar Ball Bearings — Nonholonomic Systems with Invariant Measures*, RCD, **27** (2022) 424—442.

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The rolling of a homogeneous ball over a dynamically asymmetric sphere S is introduced by Borisov, Kilin, and Mamaev (RCD, 2011)

Let I be the inertia operator of the outer sphere S. We choose the moving frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$, such that $O\vec{e}_1$, $O\vec{e}_2$, $O\vec{e}_3$ are the principal axes of inertia: $I = \mathrm{diag}(A, B, C)$. Let $\mathrm{diag}(I_i, I_i, I_i)$ and m_i be the inertia operator and the mass of the i-th ball B_i.
Then the configuration space and the kinetic energy of the problem are given by:

$$
\begin{aligned}
Q =& SO(3)^{n+1} \times (S^2)^n \{\mathrm{g}, \mathrm{g}_1, \dots, \mathrm{g}_n, \vec{\gamma}_1, \dots, \vec{\gamma}_n\},\\
T =& \frac{1}{2}\langle I\vec{\Omega}, \vec{\Omega}\rangle + \frac{1}{2}\sum_{i=1}^{n} I_i \langle \vec{\omega}_i, \vec{\omega}_i\rangle + \frac{1}{2}\sum_{i=1}^{n} m_i \langle \vec{v}_{O_i}, \vec{v}_{O_i}\rangle.
\end{aligned}
$$

Here $\vec{\gamma}_i$ is the unit vector $\vec{\gamma}_i = \frac{\overrightarrow{OO_i}}{|\overrightarrow{OO_i}|}$ determining the position O_i of the centre of i-th ball B_i and $\vec{v}_{O_i}$ is its velocity, $i = 1, \dots, n$. In cases I and II, $\vec{v}_{O_i} = (R \pm r)\dot{\vec{\gamma}}_i$ while in cases III and IV $\vec{v}_{O_1} = \pm(r - R)\dot{\vec{\gamma}}_1$.

Let us denote the contact points of the balls $\mathrm{B}_1, \dots, \mathrm{B}_n$ with the spheres S_0 and S by $A_1, ..., A_n$ and $B_1, B_2, ..., B_n$, respectively. The condition that the rolling of the balls $\mathrm{B}_1, \dots, \mathrm{B}_n$ and the sphere S are without slipping leads to the nonholonomic constraints: In cases I and II

$$\vec{v}_{O_i} = \pm r\vec{\omega}_i \times \vec{\gamma}_i, \qquad \vec{v}_{O_i} = (R \pm 2r)\vec{\omega} \times \vec{\gamma}_i \pm r\vec{\omega}_i \times \vec{\gamma}_i.$$

and in cases III and IV

$$\vec{v}_{O_1} = \pm r\vec{\omega}_1 \times \vec{\gamma}_1, \qquad \vec{v}_{O_1} = \pm(2r - R)\vec{\omega} \times \vec{\gamma}_1 \pm r\vec{\gamma}_1 \times \vec{\omega}_1.$$

The dimension of the configuration space Q is $5n + 3$. There are $4n$ independent constraints, defining a nonintegrable distribution $\mathcal{D} \subset TQ$. Therefore, the dimension of the vector subspaces of admissible velocities $\mathcal{D}_q \subset T_qQ$ is $n + 3$, $q \in Q$. The phase space of the system has the dimension $6n + 6$, which is the dimension of the bundle $\mathcal{D}$ as a submanifold of TQ.

The kinetic energy and the constraints are invariant with respect to the $SO(3)^{n+1}$–action defined by

$$(\mathrm{g}, \mathrm{g}_1, \dots, \mathrm{g}_n, \vec{\gamma}_1, \dots, \vec{\gamma}_n) \longmapsto (\mathrm{ag}, \mathrm{ag}_1\mathrm{a}_1^{-1}, \dots, \mathrm{ag}_n\mathrm{a}_n^{-1}, \mathrm{a}\vec{\gamma}_1, \dots, \mathrm{a}\vec{\gamma}_n),$$

$\mathrm{a}, \mathrm{a}_1, \dots, \mathrm{a}_n \in SO(3)$. For the coordinates in the space $(TQ)/SO(3)^{n+1}$ we can take the angular velocities and the unit position vectors in the reference frame attached to the sphere S:

$$(TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \{\vec{\Omega}, \vec{\Omega}_1, \dots, \vec{\Omega}_n, \dot{\vec{\Gamma}}_1, \dots, \dot{\vec{\Gamma}}_n, \vec{\Gamma}_1, \dots, \vec{\Gamma}_n\}.$$

In the moving reference frame $O\vec{\mathrm{e}}_1, \vec{\mathrm{e}}_2, \vec{\mathrm{e}}_3$, the constraints become:

$$\vec{V}_{O_i} = \pm r\vec{\Omega}_i \times \vec{\Gamma}_i, \qquad \vec{\Omega}_i \times \vec{\Gamma}_i = \delta\vec{\Omega} \times \vec{\Gamma}_i, \qquad i = 1, \dots, n,$$

where

$$\varepsilon = \frac{R}{2R \pm 2r} \qquad \text{and} \qquad \delta = \pm\frac{R \pm 2r}{2r} \qquad \text{(cases I and II)},$$
$$\varepsilon = \frac{R}{2R - 2r} \qquad \text{and} \qquad \delta = \frac{2r - R}{2r} \qquad \text{(cases III and IV)}.$$

Since both the kinetic energy and the constraints are invariant with respect to the $SO(3)^{n+1}$–action, the equations of motion are also $SO(3)^{n+1}$–invariant. Thus, they induce a well defined system on the reduced phase space

$$\mathcal{M} = \mathcal{D}/SO(3)^{n+1} \subset (TQ)/SO(3)^{n+1}$$

of dimension $3n+3$.

Lemma

The kinematic part of the equations of motion of the spherical ball bearing system is:

$$\dot{\vec{\Gamma}}_i = \epsilon \vec{\Gamma}_i \times \vec{\Omega}, \qquad i = 1, \dots, n.$$

Let $\vec{\mathrm{F}}_{B_i}$ and $\vec{\mathrm{F}}_{A_i}$ be the reaction forces that act on the ball B_i at the points B_i and A_i, respectively. The reaction force at the point B_i on the sphere S is then $-\vec{\mathrm{F}}_{B_i}$.

Lemma

The dynamical part of the equations of motion of the spherical ball bearing system is:

$$
\begin{aligned}
I_i\dot{\vec{\Omega}}_i &= I_i\vec{\Omega}_i \times \vec{\Omega} \pm r\vec{\Gamma}_i \times (\vec{F}_{B_i} - \vec{F}_{A_i}),\\
m_i\dot{\vec{V}}_{O_i} &= m_i\vec{V}_{O_i} \times \vec{\Omega} + \vec{F}_{B_i} + \vec{F}_{A_i}, \qquad\qquad i = 1, ..., n\\
I\dot{\vec{\Omega}} &= I\vec{\Omega} \times \vec{\Omega} \mp 2r\delta \sum_{i=1}^{n} \vec{\Gamma}_i \times \vec{F}_{B_i}.
\end{aligned}
$$

Proposition

We have following first integrals

$$
\begin{aligned}
\langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle &= c_i = const, \qquad i = 1, ..., n\\
\langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle &= \gamma_{ij} = const, \qquad 1 \leqslant i < j \leqslant n.
\end{aligned}
$$

So, the centers O_i of the homogeneous balls B_i are in rest in relation to each other.

The reduced system

The reduced phase space $\mathcal{M} = \mathcal{D}/SO(3)^{n+1}$ is foliated on $2n+3$–dimensional invariant varieties

$$\mathcal{M}_c : \qquad \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = const, \qquad i = 1, ..., n.$$

On the invariant variety $\mathcal{M}_c$, the vector-functions $\vec{\Omega}_i$ can be uniquely expressed as functions of $\vec{\Omega}$, $\vec{\Gamma}_i$:

$$\vec{\Omega}_i = c_i \vec{\Gamma}_i + \delta \vec{\Omega} - \delta \langle \vec{\Gamma}_i, \vec{\Omega} \rangle \vec{\Gamma}_i.$$

Whence, $\vec{\Omega}$ determines all velocities of the system on $\mathcal{M}_c$ and $\mathcal{M}_c$ is diffeomorphic to the *second reduced phase space*

$$\mathcal{N} = \mathbb{R}^3 \times \left(S^2\right)^n \{\vec{\Omega}, \vec{\Gamma}_1, \dots, \vec{\Gamma}_n\}.$$

As a result we obtain the following diagram

$$
\begin{array}{ccccc}
\mathcal{D}_c & \hookrightarrow & \mathcal{D} & \hookrightarrow & TQ = (TSO(3))^{n+1} \times (TS^2)^n \\
\big\downarrow /SO(3)^{n+1} & & \big\downarrow /SO(3)^{n+1} & & \big\downarrow /SO(3)^{n+1} \\
\mathcal{M}_c & \hookrightarrow & \mathcal{M} & \hookrightarrow & (TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \\
 & \searrow^{\pi_c}_{\cong} & & & \big\downarrow \pi \\
 & & & & \mathcal{N} = \mathbb{R}^3 \times \left(S^2\right)^n
\end{array}
$$

We set

$$
\vec{M} = \mathrm{I}\vec{\Omega} = I\vec{\Omega} + \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\vec{\Omega} - \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\langle \vec{\Gamma}_i, \vec{\Omega}\rangle \vec{\Gamma}_i,
$$

$$
\vec{N} = \delta \sum_{i=1}^{n} I_i c_i \vec{\Gamma}_i .
$$

We refer to I as the modified inertia operator.

Theorem

The reduction of the spherical ball bearing problem to $\mathcal{M}_c \cong \mathcal{N}$ is described by the equations

$$\frac{d}{dt}\vec{M} = \vec{M} \times \vec{\Omega} + (1-\varepsilon)\vec{N} \times \vec{\Omega},$$
$$\frac{d}{dt}\vec{\Gamma}_i = \varepsilon\vec{\Gamma}_i \times \vec{\Omega}, \qquad i = 1, \dots, n.$$

The kinetic energy of the system takes the form

$$T = \frac{1}{2}\langle \vec{M}, \vec{\Omega} \rangle + \frac{1}{2}\sum_{i=1}^{n} I_i c_i^2.$$

Also, since

$$\frac{d}{dt}\vec{N} = \varepsilon\vec{N} \times \vec{\Omega},$$

the first equation is equivalent to

$$\frac{d}{dt}(\vec{M} + \vec{N}) = (\vec{M} + \vec{N}) \times \vec{\Omega}.$$

Theorem

For arbitrary values of parameters c_i, the reduced system has the invariant measure

$$\mu(\vec{\Gamma}_1, \dots, \vec{\Gamma}_n) d\Omega \wedge \sigma_1 \wedge \cdots \wedge \sigma_n, \qquad \mu = \sqrt{\det(\mathrm{I})},$$

where $d\Omega$ and σ_i are the standard measures on $\mathbb{R}^3\{\vec{\Omega}\}$ and $S^2\{\vec{\Gamma}_i\}$, $i = 1, \dots, n$.

Proposition

The system always has the following first integrals

$$F_1 = \frac{1}{2}\langle \vec{M}, \vec{\Omega} \rangle, \quad F_2 = \langle \vec{M} + \vec{N}, \vec{M} + \vec{N} \rangle, \quad F_{ij} = \langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle, \quad 1 \le i < j \le n.$$

Thus, in the special case $n = 1$, we have the 5-dimensional phase space $\mathcal{N} = \mathbb{R}^3 \times S^2\{\vec{\Omega}, \vec{\Gamma}_1\}$, and the system has two first integrals and an invariant measure. For the integrability, one needs to find a third independent first integral.

Spherical support system and ε–modified L+R systems

If we set $\varepsilon = 1$ in the system, we obtain the equation of the spherical support system introduced by Fedorov. The system describes the rolling without slipping of a dynamically nonsymmetric sphere S over n homogeneous balls $B_1, \dots, B_n$ of possibly different radii, but with fixed centers. It is an example of a class of nonhamiltonian L+R systems on Lie groups with an invariant measure.

On the other hand, if we set $\vec{N} = 0$, we obtain an example ε–modified L+R system.

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System with one homogeneous ball

We proceed with the case $n = 1$. To simplify notation, we denote $\vec{\Gamma}_1$ by $\vec{\Gamma}$ and set

$$D = \delta^2(I_1 + m_1 r^2), \qquad d = \delta I_1 c_1, \qquad L = \langle \vec{\Omega}, \vec{\Gamma} \rangle,$$
$$\vec{\mathrm{M}} = \vec{M} + \vec{N} = \vec{M} + d\vec{\Gamma} = I\vec{\Omega} + D\vec{\Omega} + (d - DL)\vec{\Gamma}.$$

The reduced system, the operator I, and its determinant now read

$$\dot{\vec{\mathrm{M}}} = \vec{\mathrm{M}} \times \vec{\Omega}, \qquad \dot{\vec{\Gamma}} = \varepsilon \vec{\Gamma} \times \vec{\Omega},$$

$$\mathrm{I} = \mathbb{I} + D\mathrm{E} - D\vec{\Gamma} \otimes \vec{\Gamma} = \begin{pmatrix} A + D - D\Gamma_1^2 & -D\Gamma_1\Gamma_2 & -D\Gamma_1\Gamma_3 \\ -D\Gamma_1\Gamma_2 & B + D - D\Gamma_2^2 & -D\Gamma_2\Gamma_3 \\ -D\Gamma_1\Gamma_3 & -D\Gamma_2\Gamma_3 & C + D - D\Gamma_3^2 \end{pmatrix},$$

$$\det(\mathrm{I}) = (A + D)(B + D)(C + D)\big(1 - D\big(\frac{\Gamma_1^2}{A + D} + \frac{\Gamma_2^2}{B + D} + \frac{\Gamma_3^2}{C + D}\big)\big).$$

The first integrable case (generic $\mathbb{I}$, $\varepsilon = -1$)

Condition $\varepsilon = -1$ corresponds to configuration III for $2r = 3R$. We get the following statement.

Theorem

The spherical ball bearings problem (17) *in the configuration III, when $2r = 3R$, i.e., the radius of the moving sphere* S *is twice the radius of the fixed sphere* S_0, *is integrable. The third integral is*

$$F_3 = (B+C-A+D)M_1\Gamma_1 + (A+C-B+D)M_2\Gamma_2 + (A+B-C+D)M_3\Gamma_3.$$

For $d = 0$, the integral F_3 reduces to the one found by Borisov and Fedorov for the rolling of a Chaplygin ball over a sphere.

The second integrable case ($B = C$, generic ε)

Theorem

The spherical ball bearings problem (17) *for $B = C$ is integrable for all ε. Along with F_1 and F_2, the system has two additional, nonalgebraic first integrals F_3 and F_4:*

$$F_{3,4} = \big(\pm\sqrt{D(A-C)}F + DG - dC\big)\exp(\pm(1-\varepsilon)\sqrt{D(A-C)}\Phi).$$

Note that the product of the two nonalgebraic first integrals

$$\begin{aligned} F_3F_4 =& (DG - dC)^2 - D(A-C)F^2 \\ =& D^2G^2 - 2dCDG + d^2C^2 - D(A-C) \\ & \big(C(A+D)\Omega_1^2 + D(A-C)\Omega_1^2\Gamma_1^2\big) \end{aligned}$$

is an affine combination of F_1 and F_2:

$$F_3F_4 = CD(C+D)\langle\vec{M},\vec{\Omega}\rangle - CD\langle\vec{\mathrm{M}},\vec{\mathrm{M}}\rangle - C(C+D)d^2.$$

Rubber Chaplygin ball, gyroscopic Chaplygin ball

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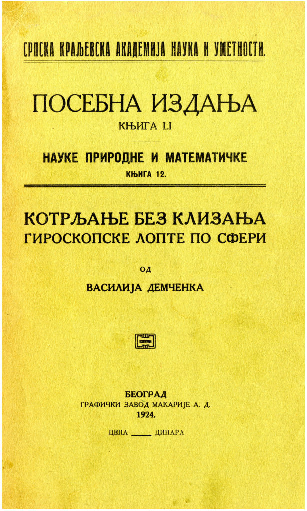
Demchenko PhD thesis under supervision of A. Bilimović, M. Milanković and M. Petrović



СРПСКА КРАЉЕВСКА АКАДЕМИЈА НАУКА И УМЕТНОСТИ.

ПОСЕБНА ИЗДАЊА
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1924.

ЦЕНА ____ ДИНАРА

Rubber and gyroscopic Chaplygin ball II

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Thank you!!!

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