Multidimensional solitons

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(1) Introduction

Formation of ***localized structures*** in the form of ***solitons*** in many physical settings is accounted for by the interplay of nonlinear ***self-attraction*** (alias ***self-focusing***) of physical fields (e.g., electromagnetic waves in photonics, or macroscopic wave functions, alias ***matter waves***, in ***Bose-Einstein condensates***, **BECs**) and basic linear effects, such as ***dispersion*** or ***diffraction*.**

Arguably, the most important model which creates solitons is the ***nonlinear Schrödinger*** **(NLS)** ***equation:***

$$i\frac{\partial\Psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}-|\Psi|^2\Psi.$$

It generates a family of bright-soliton solutions with two free parameters - amplitude $\boldsymbol{\eta}$ and velocity $\boldsymbol{c}$:

$$\Psi(x,t)=e^{icx+i\left(\eta^2-c^2\right)t/2}\frac{\eta}{\cosh(\eta(x-ct))}\,.$$

The ***exact integrability*** of this equation, and of its counterpart with the ***self-repulsive*** (***defocusing***) nonlinearity,

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} \pm |\Psi|^2\,\Psi,$$

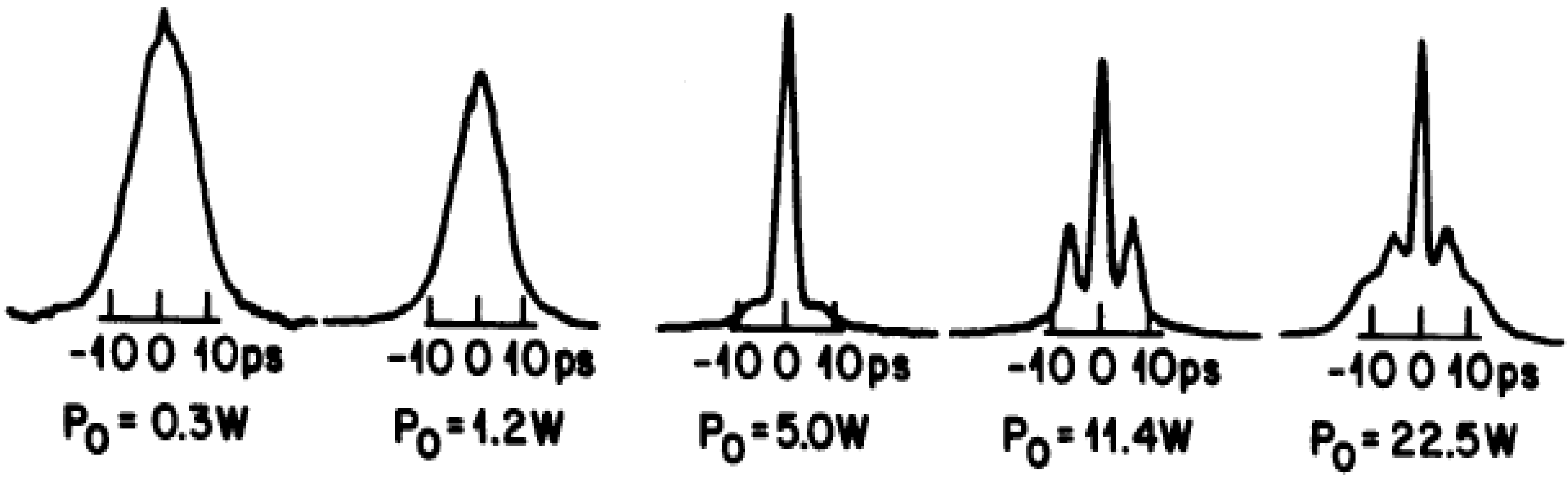
by means of the ***inverse-scattering transform***, had been discovered in the classical work by **V.E. Zakharov** and **A.B. Shabat** (originally published in Russian),

V. E. Zakharov and A. B. Shabat, *Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media*,

J. Exp. Theor. Phys. **34**, 62-69 (1972).

Solitons of the NLS type in ***nonlinear optical fibers*** were predicted by Hasegawa and Tappert in 1973 (Appl. Phys. Lett. **23**, 142-144), and experimentally created by Mollenauer, Stolen and Gordon in 1980 (Phys. Rev. Lett. **45**, 1095-1098).

They also observed two-soliton bound states (*breathers*):

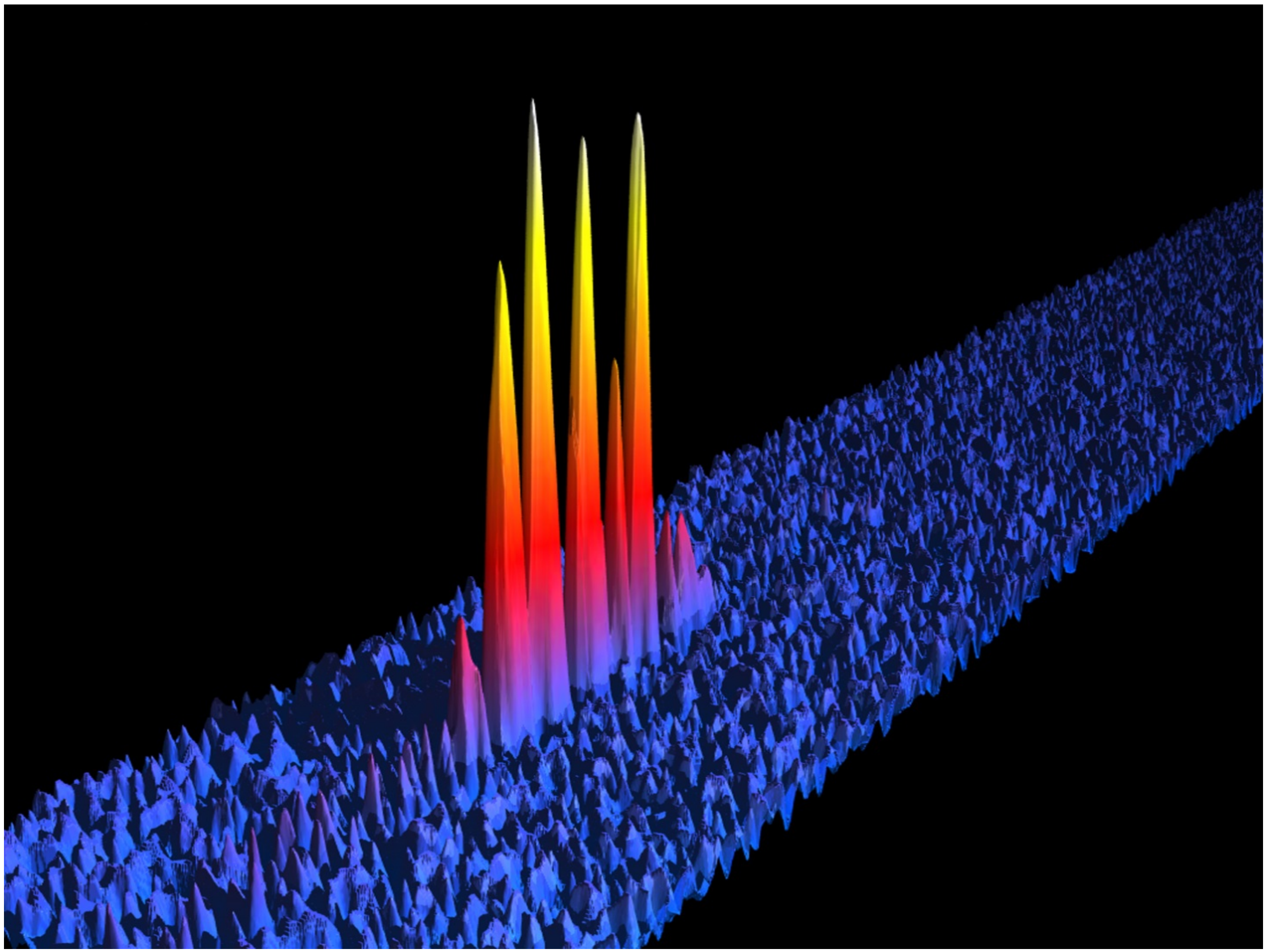


Another famous realization of solitons was demonstrated in **BEC** loaded in nearly one-dimensional (“cigar-shaped”) trapping potentials. Such ***bright matter-wave solitons*** were first created in the condensate of ^{7}Li atoms:

K.E. Strecker, G.B. Partridge, A.G. Truscott, and R. G. Hulet, Nature **417**, 150 (2002);

L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L.D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).

The famous experimental picture of the atomic density distribution in a chain of **7 matter-wave solitons** with unequal amplitudes in ^{7}Li (from the work of R. Hulet *et al.*):



(2) The subject of the talk: multidimensional solitons

A more challenging objective is to **predict** and **create experimentally** **two**- and **three-dimensional** (**2D** and **3D**) solitons. On the contrary to their **1D** counterparts, which are, normally, ***stable***, the ubiquitous ***cubic self-focusing*** always makes **2D** and **3D** solitons ***unstable***, as ***exactly the same*** self-interaction gives rise to fundamental ***destabilizing effects***: the ***critical collapse*** in **2D**, and ***supercritical collapse*** in **3D** (i.e., ***catastrophic self-compression*** of the wave function, which leads to ***formation of a singularity*** after a finite evolution time).

First, let us look at **2D** solitons produced by the **NLS** equation with the **self-focusing cubic nonlinearity**,

$$\boldsymbol{iu_t + (1/2)(u_{xx} + u_{yy}) + |u|^2 u = 0.}$$

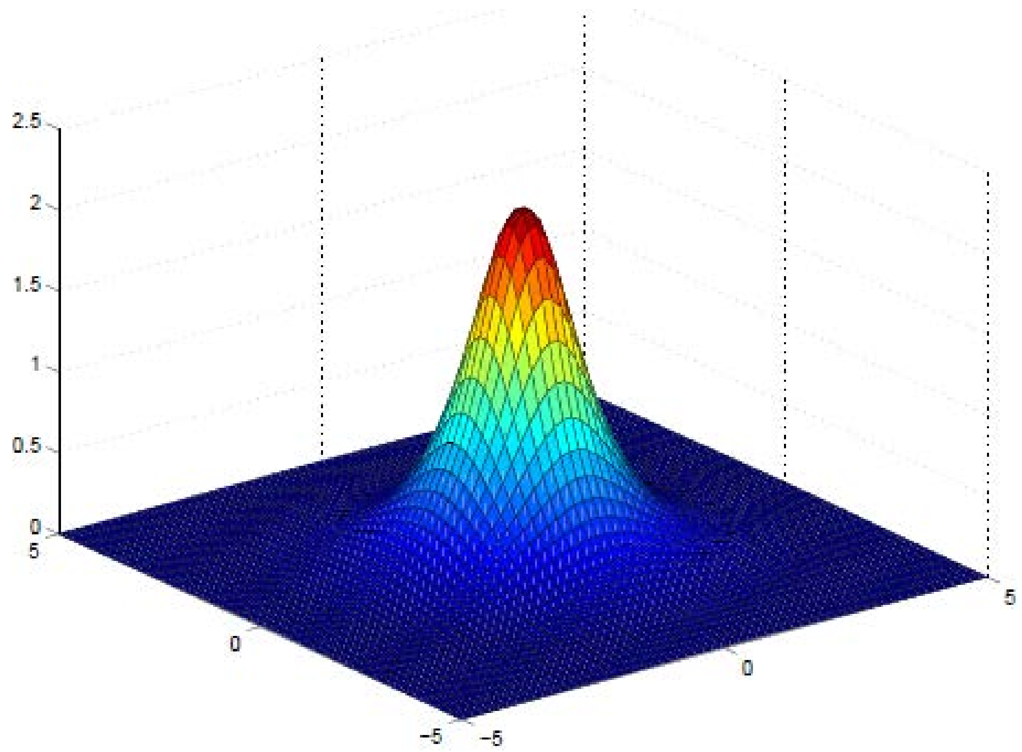
Soliton solutions with integer **vorticity *S*** and **chemical potential** μ are looked for, in the polar coordinates $(\boldsymbol{r},\boldsymbol{\theta})$, as

$$\boldsymbol{u = \exp(-i\mu t + iS\theta)\, U_S(r).}$$

The real radial wave function $U_S(r)$ satisfies the equation

$$\mu U_S + \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{S^2}{r^2}\right)U_S + U_S^3 = 0.$$

The profile of $U_0(r)$ for a **fundamental (zero-vorticity, S = 0) 2D soliton** [alias the ***Townes' soliton***, R.Y. Chiao, E. Garmire & C. H. Townes, Phys. Rev. Lett . **13**, 479 (1964)]:



The families of the Townes' solitons (and their vortex counterparts) are ***degenerate***: due to the specific ***conformal symmetry*** of the **2D NLS** equation with the cubic nonlinearity, the norm of each family (with ***S* = 0,1,2**, *etc*.),

$$N_S = 2\pi \int_0^\infty U_S^2(r;\mu) r dr,$$

takes a ***single value***, which ***does not*** depend on the soliton's ***chemical potential***, ***μ***. In particular,

for $S = 0$ (the fundamental Townes' solitons), $N_0 \approx \mathbf{5.85}$.

For $S \geq 1$, the norm is accurately approximated by

$$N_S = 2\pi \int_0^{\infty} U_S^2(r;\mu) r dr \approx 4\sqrt{3}\,\pi S,$$

as demonstrated in

PHYSICAL REVIEW A **94**, 053611 (2016)

Stable giant vortex annuli in microwave-coupled atomic condensates

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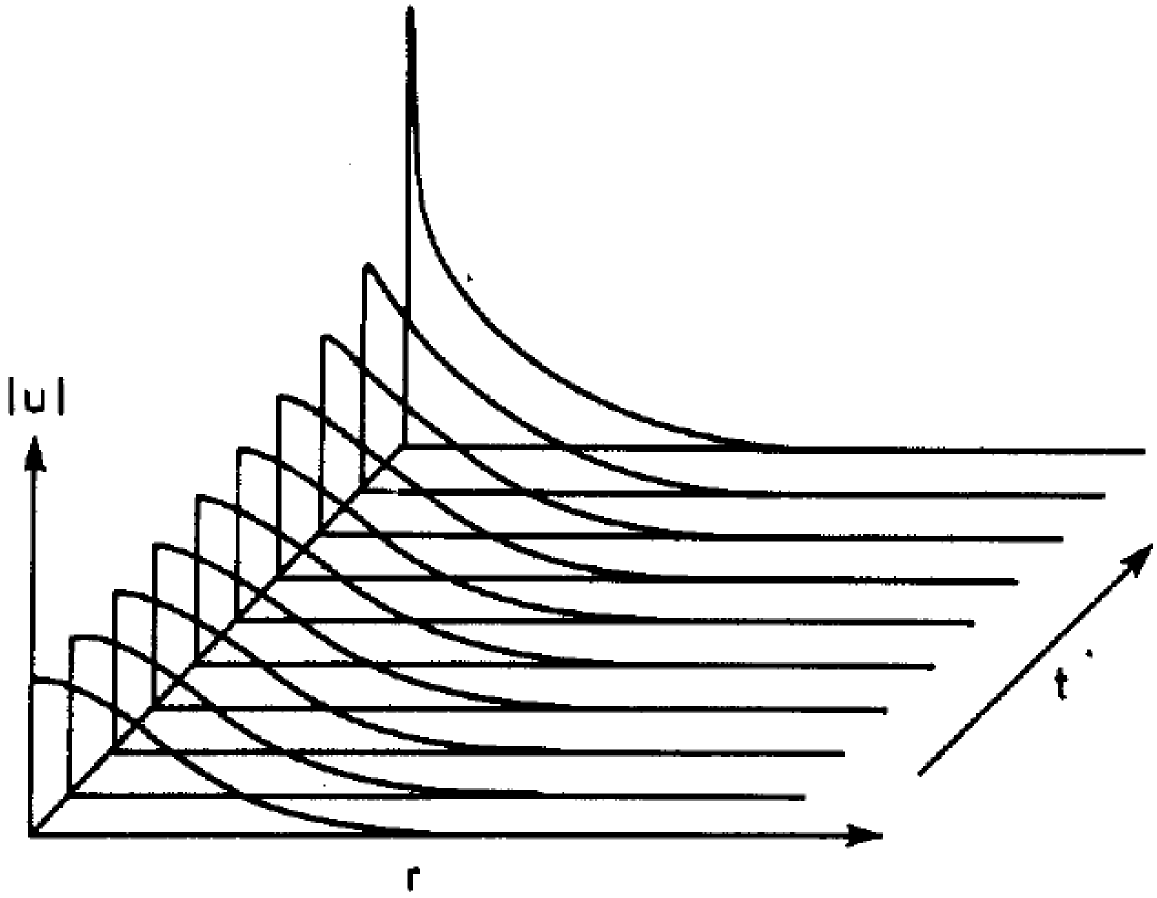
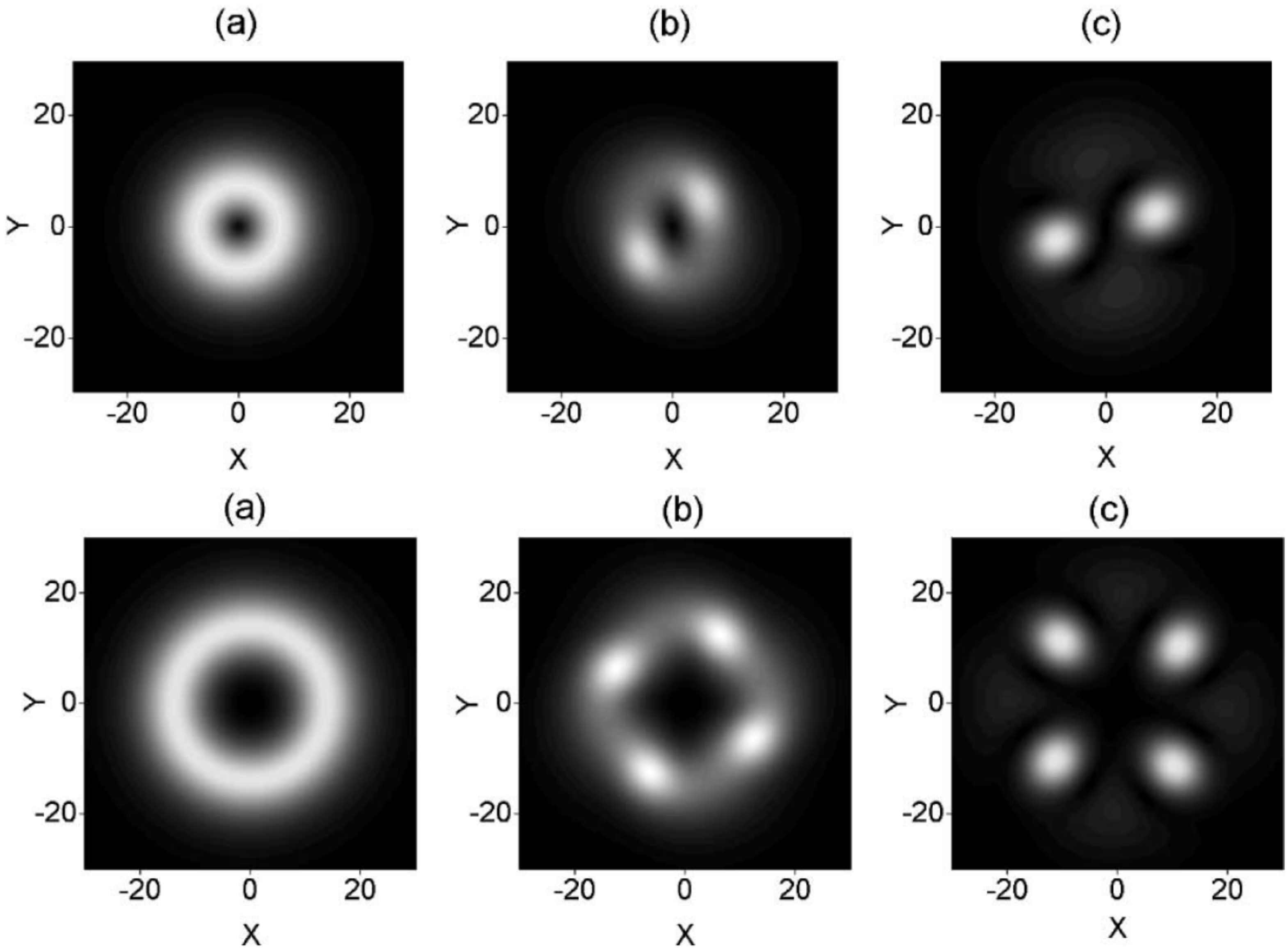
(Received 3 March 2016; revised manuscript received 17 October 2016; published 14 November 2016)

The numerically simulated development of the ***collapse*** of the fundamental Townes soliton in **2D** (the evolution of the radial cross section of the collapsing soliton towards the formation of a ***singularity***):



Examples of the ***spontaneous splitting*** of ***unstable*** **2D** ***vortex solitons*** with **vorticities *S* = 1** and ***S* = 2:**



The norm of the ***Townes' soliton*** is a ***threshold value*** which separates ***collapsing*** ($N > N_0$) and ***decaying*** ($N < N_0$) localized solutions of the **2D NLS** equation.

A ***stabilizing mechanism***, added to the **2D NLS** equation, may act by letting the norm of **2D** solitons take values $N < N_0$ (but ***avoiding the decay***), so that they ***cannot*** undergo the ***collapse***, as their norm falls below the threshold value necessary to initiate the collapse.
Stabilization of ***vortex solitons*** is a ***still more challenging problem***.

However, in the **3D NLS** equation, the situation is more difficult, as the collapse is ***supercritical***, with ***zero threshold***. This fact suggests that **3D** solitons may be made ***metastable***, in the best case, as ***the collapse remains possible***.

Thus, ***stabilization*** is the ***most important*** issue for **multidimensional solitons** (alias **spatiotemporal solitons**, in terms of optics). A review summarizing theoretical and experimental findings in this area:

Nature Reviews Physics 1, 185-197 (2019)

Frontiers in multidimensional self-trapping of nonlinear fields and matter

Yaroslav V. Kartashov[1,2], *Gregory E. Astrakharchik*[3], *Boris A. Malomed*[4,5,6,7] *and Lluis Torner*[1,8]*

Another review article, specifically focused on **2D** and **3D** solitons with ***embedded vorticity*** (which is one of central topics of the present talk):

Physica D 399 (2019) 108–137

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Invited Review Article

(INVITED) Vortex solitons: Old results and new perspectives

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A recent review article summarizes theoretical and experimental findings for multidimensional solitons in ***dissipative media***:

Chaos, Solitons and Fractals 163 (2022) 112526

ELSEVIER

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Multidimensional dissipative solitons and solitary vortices

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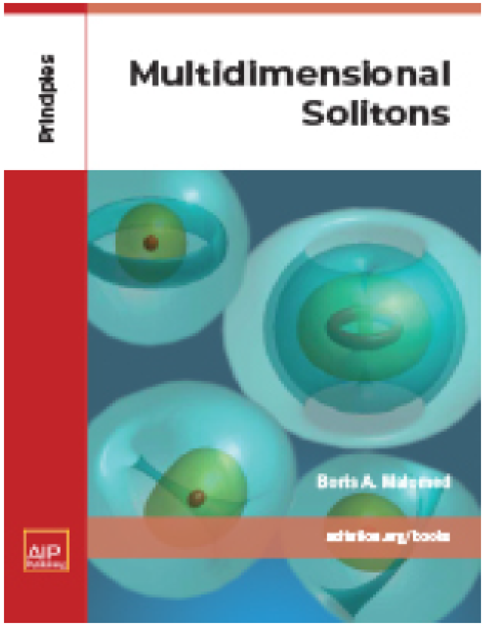
All the results, theoretical and experimental ones, are presented in a systematic form in a **new book** (published by the American Institute of Physics, Melville, NY, 2022):



Full . September 2022

Multidimensional Solitons

Author Boris A. Malomed

Publisher: AIP Publishing

Pages: 406

Copyright year: 2022

As the **stability** is the main issue for **2D** and **3D** settings, the rest of the talk is structured according to basic mechanisms which provide for **stabilization** of the multidimensional solitons:

(3) Relatively old findings: **Stable** **2D** and **3D vortex solitons** in models with the **cubic-quintic** (**CQ**) nonlinearity (in a brief form).

(4) A relatively new model: **Stable** **2D** and **3D** composite solitons in **two-component spin-orbit (SO)-coupled BEC**.

(5) **Latest** theoretical and experimental results: the prediction and creation of **3D** and **2D** matter-wave solitons ("***quantum droplets***") stabilized by **quantum fluctuations**.

(6) Conclusions.

(3) 2D and 3D Systems with the cubic-quintic nonlinearity

The stabilization of **2D** and **3D** fundamental and vortical solitons can be provided by a combination of ***competing self-focusing cubic*** and ***self-defocusing quintic*** nonlinear terms.

In optics, the **3D NLS** equation can be written as the equation governing the ***spatiotemporal evolution*** of the electromagnetic field:

$$i\frac{\partial u}{\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial \tau^2}+|u|^2 u-|u|^4 u=0.$$

Stationary solutions with integer vorticity, $S \geq 0$, and propagation constant (wavenumber) k, are looked for in the cylindrical spatiotemporal coordinates as

$$u(x,y,\tau,z)=U(r,\tau)\exp(ikz+iS\theta),\ \ r\equiv\sqrt{x^2+y^2},$$

with $U(r,\tau)$ satisfying the stationary equation:

$$\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{S^2}{r^2}U+\frac{\partial^2 U}{\partial \tau^2}\right)+U^3-U^5=kR.$$

The 2D reduction of the **3D** equations implies dropping the term $\partial^2 U/\partial\tau^2$.

The stability of ***fundamental solitons*** (***S*** **= 0**) in the framework of this equation is obvious. A nontrivial problem is the ***stability of vortex solitons*** against splitting by azimuthal perturbations.

For **2D** ***vortex solitons***, the ***stability*** was first reported in

J. Opt. Soc. Am. B/Vol. 14, No. 8/August 1997 M. Quiroga-Teixeiro and H. Michinel

Stable azimuthal stationary state in quintic nonlinear optical media

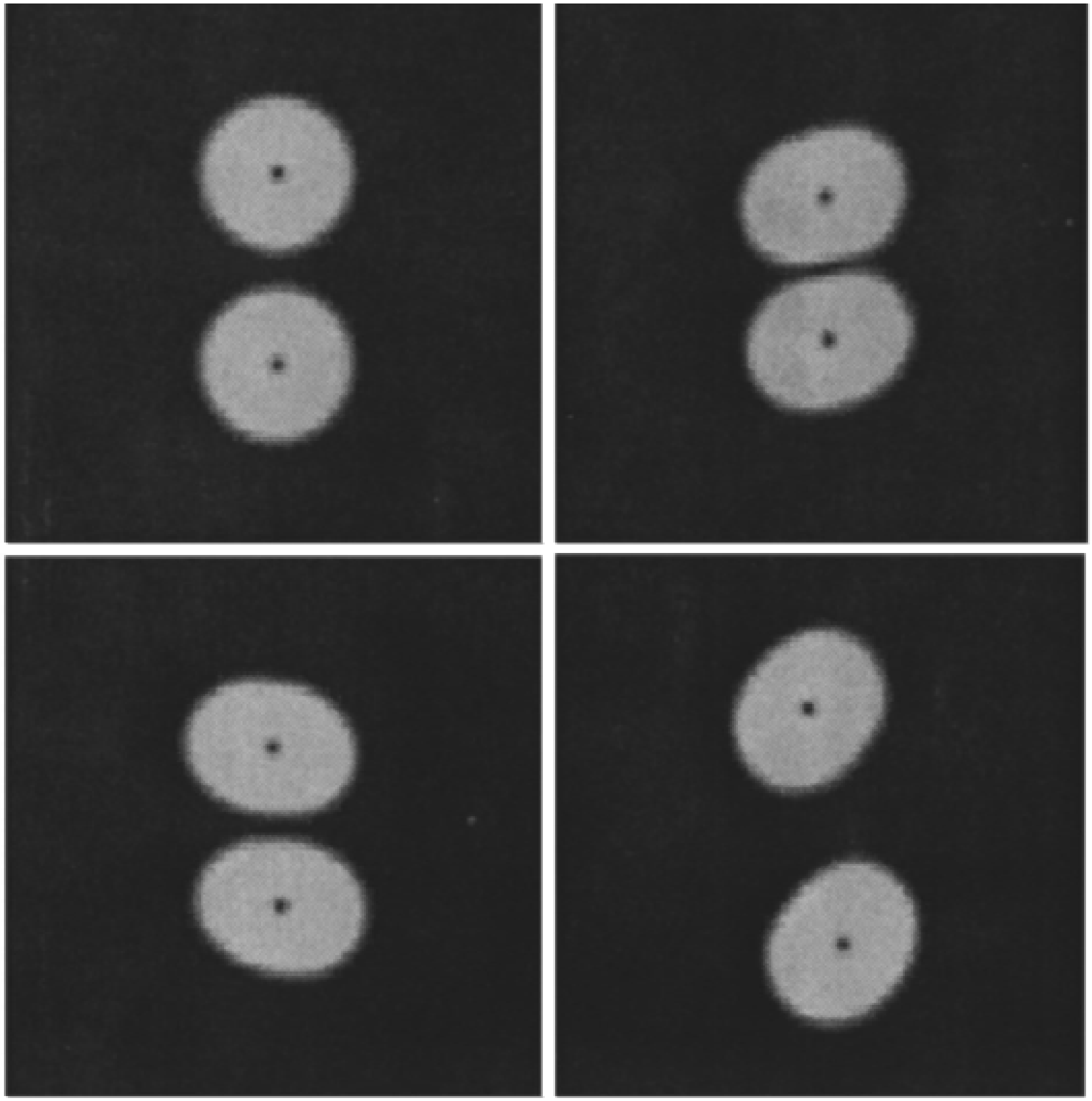
M. Quiroga-Teixeiro

Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-412 96, Göteborg, Sweden

H. Michinel

Departamento de Física Aplicada, Escola Universitaria de Óptica, Universidade de Santiago de Compostela, E-157 06, Santiago de Compostela, Galicia, Spain

An example of the **dynamical stability** and **elastic collision** of **2D vortex solitons** with ***S* = 1**:



Experimentally, the stability of **(2+1)D** ***fundamental*** (***S*** **= 0**) solitons in an optical ***cubic-quintic*** medium was demonstrated relatively recently:

PRL **110,** 013901 (2013) PHYSICAL REVIEW LETTERS week ending 4 JANUARY 2013

Robust Two-Dimensional Spatial Solitons in Liquid Carbon Disulfide

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Georges Boudebs, Hervé Leblond, and Vladimir Skarka
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(Received 29 March 2012; published 2 January 2013)

The excitation of near-infrared $(2+1)$D solitons in liquid carbon disulfide is demonstrated due to the simultaneous contribution of the third- and fifth-order susceptibilities. Solitons propagating free from diffraction for more than 10 Rayleigh lengths although damped, were observed to support the proposed soliton behavior. Numerical calculations using a nonlinear Schrödinger-type equation were also performed.

Fully stable ***vortex solitons*** have not yet been created in the experiment. However, ***quasi-stable*** ones have been reported in

PHYSICAL REVIEW A **93**, 013840 (2016)

Robust self-trapping of vortex beams in a saturable optical medium

Albert S. Reyna,[1,*] Georges Boudebs,[2] Boris A. Malomed,[1,†] and Cid B. de Araújo[1]
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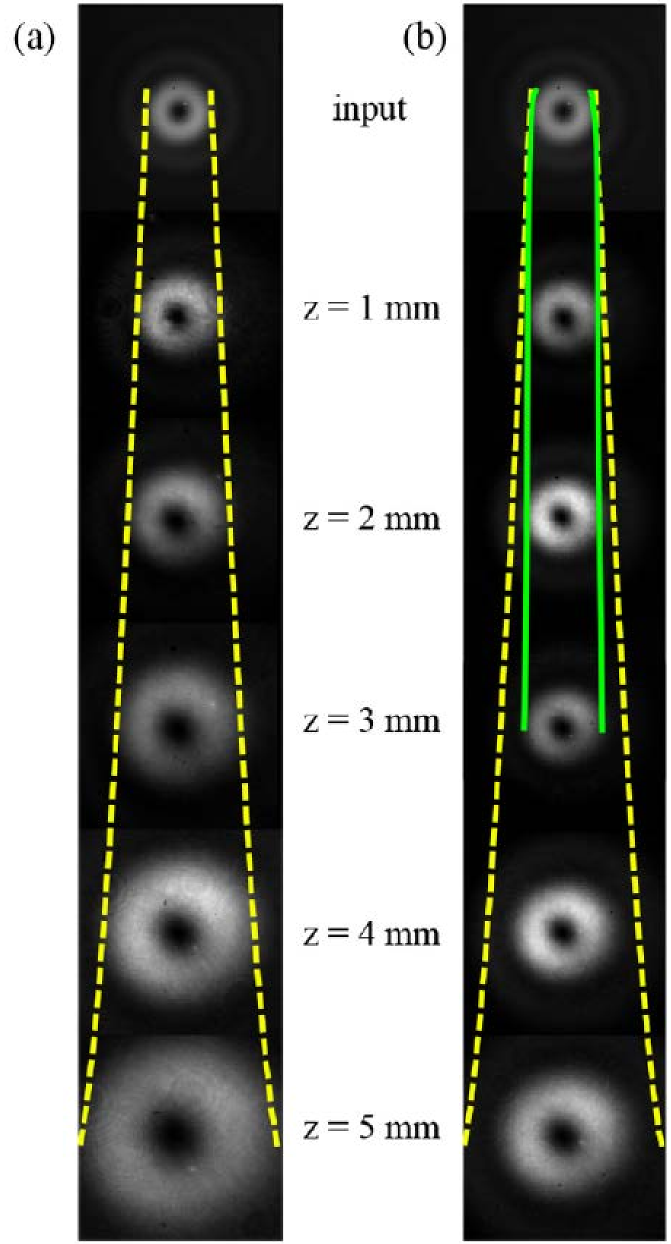
[2]*LUNAM Université, Université d'Angers, Laboratoire de Photonique d'Angers, EA 4464, 49045 Angers, France*
(Received 28 March 2015; published 21 January 2016)

The vortex soliton starts expansion due to nonlinear losses (multiphoton absorption).



A challenging problem is to construct **3D** vortex solitons in the cubic-quintic medium, and analyze their **stability**. Theoretically, this was done long ago in:

VOLUME 88, NUMBER 7 PHYSICAL REVIEW LETTERS 18 FEBRUARY 2002

Stable Spinning Optical Solitons in Three Dimensions

D. Mihalache,[1,2,5] D. Mazilu,[1,2] L.-C. Crasovan,[1,5] I. Towers,[3,4] A. V. Buryak,[3] B. A. Malomed,[4] L. Torner,[5] J. P. Torres,[5] and F. Lederer[2]

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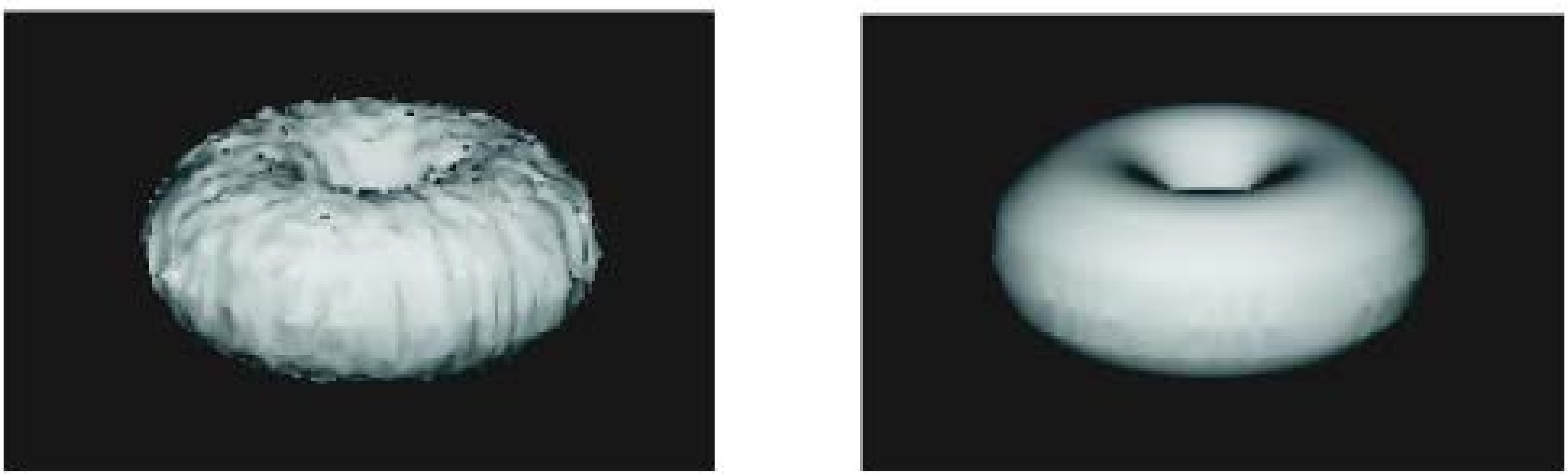
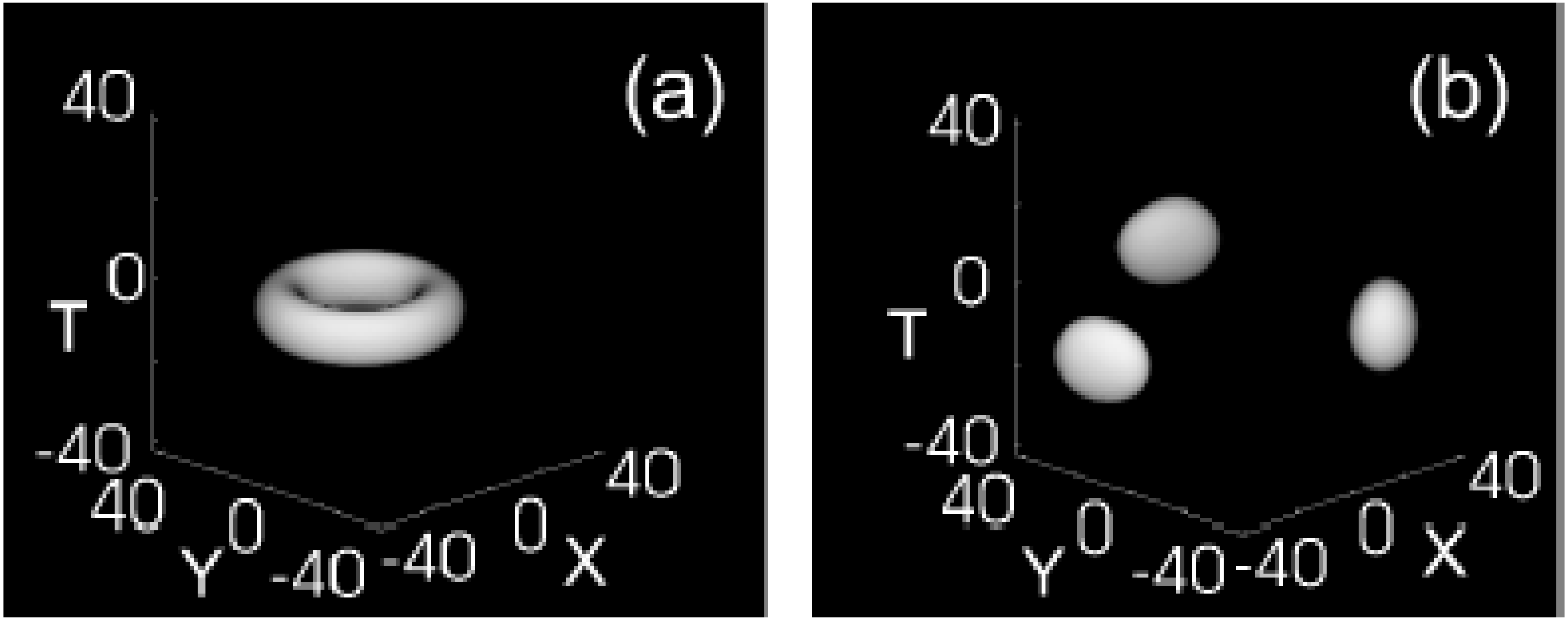
[2]*Institute of Solid State Theory and Theoretical Optics, Friedrich-Schiller University Jena, Max-Wien-Platz 1, D-07743, Jena, Germany*
[3]*School of Mathematics and Statistics, University of New South Wales at ADFA, Canberra, ACT 2600, Australia*
[4]*Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[5]*Department of Signal Theory and Communications, Universitat Politecnica de Catalunya, ES 08034 Barcelona, Spain*
(Received 2 July 2001; published 4 February 2002)

An example of numerically simulated ***recovery*** of a strongly perturbed **3D** ***stable*** soliton with intrinsic vorticity ***S* = 1** (***doughnut***):

(a) (b)



An example of the ***instability*** of the **3D doughnut soliton** with vorticity ***S* = 2**, *viz*., ***splitting*** in ***three fragments***:



(4) Novel results: Stable two- and three-dimensional composite solitons in spin-orbit (SO)-coupled self-attractive BEC

(4a) Introduction and objectives

The concept of ***emulation*** (alias ***simulation***) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (***matter waves***) and **photonics** (***optical waves***), has drawn a great deal of interest:

P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, Rep. Prog. Phys. **75**, 082401 (2012).

An important topic which is enabled by this approach: the emulation of **spin-orbit** (**SO**) **interactions** in semiconductors, such as those accounted for by the ***Rashba*** and ***Dresselhaus*** Hamiltonians. It is performed by ***mapping*** the spinor wave function of electrons (***fermions***) into the pseudo-spinor (two-component) ***bosonic*** wave function of a binary **BEC** gas:

Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, Nature **471**, 83 (2011);

Y. Zhang, L. Mao, and C. Zhang, Phys. Rev. Lett. **108**, 035302 (2012);

A review: *H. Zhai*, Rep. Prog. Phys. **78,** 026001 (2015).

(4b) The model

The system of **2D** Gross-Pitaevskii (**GP**) equations for the two-component wave function **($\boldsymbol{\Phi_+}$, $\boldsymbol{\Phi_-}$)** of the binary **BEC** coupled by the **SO** terms with strength $\boldsymbol{\lambda}$ and coefficient of the cubic self-attraction **≡ 1**:

$$i\frac{\partial\phi_+}{\partial t} = -\frac{1}{2}\nabla^2\phi_+ - \left|\phi_+\right|^2\phi_+ + \lambda\left(\frac{\partial\phi_-}{\partial x} - i\frac{\partial\phi_-}{\partial y}\right),$$

$$i\frac{\partial\phi_-}{\partial t} = -\frac{1}{2}\nabla^2\phi_- - \left|\phi_-\right|^2\phi_- - \lambda\left(\frac{\partial\phi_+}{\partial x} + i\frac{\partial\phi_+}{\partial y}\right).$$

(4c) References. **Basic results are presented here for 2D solitons as per the following papers:**

PHYSICAL REVIEW E **89**, 032920 (2014)

Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space

Hidetsugu Sakaguchi and Ben Li
Department of Applied Science for Electronics and Materials, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga, Fukuoka 816-8580, Japan

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(Received 12 December 2013; published 26 March 2014)

PHYSICAL REVIEW E **94**, 032202 (2016)

Vortex solitons in two-dimensional spin-orbit coupled Bose-Einstein condensates: Effects of the Rashba-Dresselhaus coupling and Zeeman splitting

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Department of Physical Chemistry, University of the Basque Country UPV-EHU, 48940 Bilbao, Spain and IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

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(Received 18 February 2016; published 2 September 2016)

(4d) Semi-vortex states

The coupled **GP** equations admit a family of solutions for ***semi-vortices***, with vorticities $\boldsymbol{m_+ = 0}$ in one component, and $\boldsymbol{m_- = 1}$ in the other. The ***exact ansatz*** for these solutions, compatible with the underlying equations ($\boldsymbol{\mu < 0}$ is the chemical potential):

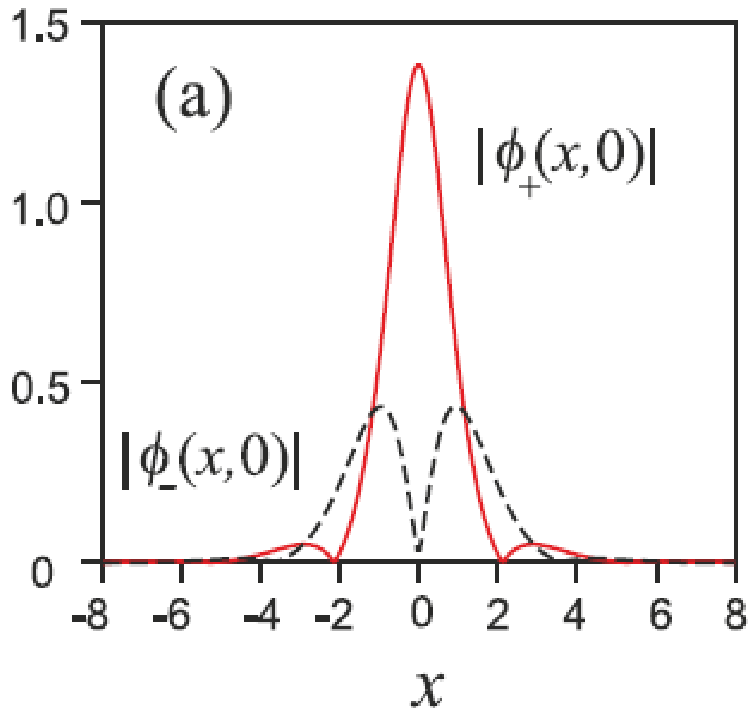
$$\phi_+(x,y,t) = \exp(-i\mu t)\, f_+(r),$$

$$\phi_-(x,y,t) = \exp(-i\mu t + i\theta)\, r f_-(r),$$

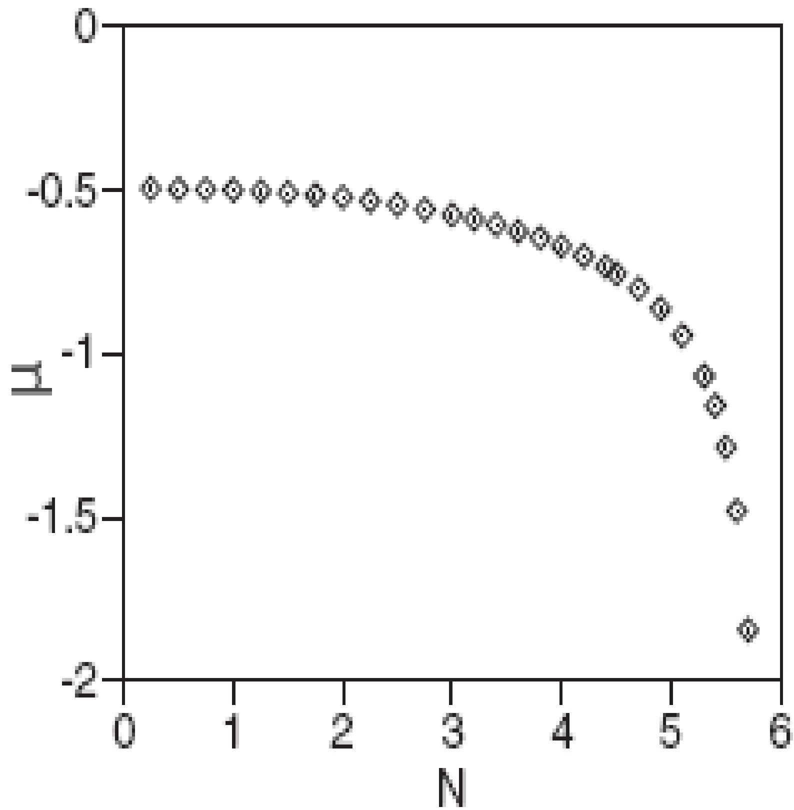
with $f_\pm(r)$ taking finite values at $r = 0$

and decaying $\sim \exp(-\sqrt{-2\mu}\, r)$ at $r \to \infty$.

A numerically generated cross-section (along ***y* = 0**) of the two components, **A ≡ |*φ*$_{\pm}$|**, for a ***stable semi-vortex***:



The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that **(1)** the norm of the semi-vortex indeed falls ***below the threshold value*** necessary for the onset of the collapse: $\boldsymbol{N(\mu) < N_{thr} \equiv N(\mu \to -\infty) \approx 5.85}$; **(2)** there is ***no finite minimum value*** of the norm necessary for the existence of the semi-vortex; **(3)** the dependence satisfies the ***Vakhitov-Kolokolov*** criterion, $\boldsymbol{d\mu/dN < 0}$, which is a necessary condition for the stability:



(4e) Mixed modes

The extended system of the **2D** Gross-Pitaevskii equations for the two-component wave function **(Φ_+, Φ_-),** which includes both the **linear SO** coupling and **nonlinear cross-interaction** with relative strength **$\gamma > 0$**:

$$i\frac{\partial \phi_+}{\partial t} = -\frac{1}{2}\nabla^2\phi_+ - \left(\left|\phi_+\right|^2 + \gamma\left|\phi_-\right|^2\right)\phi_+ + \lambda\left(\frac{\partial \phi_-}{\partial x} - i\frac{\partial \phi_-}{\partial y}\right),$$

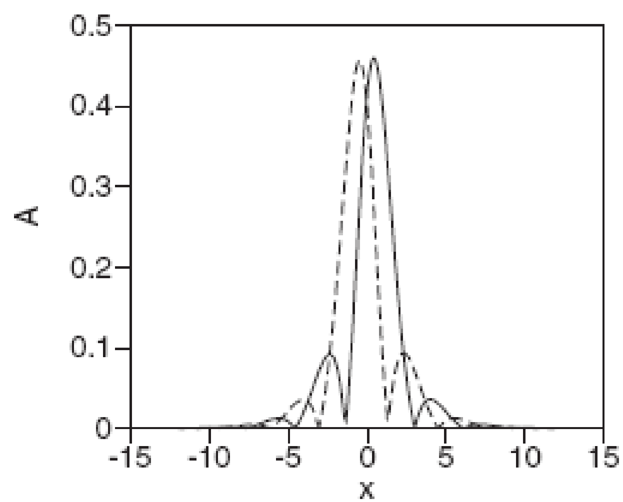
$$i\frac{\partial \phi_-}{\partial t} = -\frac{1}{2}\nabla^2\phi_- - \left(\left|\phi_-\right|^2 + \gamma\left|\phi_+\right|^2\right)\phi_- - \lambda\left(\frac{\partial \phi_+}{\partial x} + i\frac{\partial \phi_+}{\partial y}\right).$$

In the presence of the nonlinear cross-coupling, the semi-vortices remain stable at $\gamma < 1$, and become **unstable** at $\gamma < 1$. In that case, another class of **stable** localized states can be constructed in the form of **mixed modes**, so called because they **mix** fundamental and vortical terms in each component, namely, $m_1 = (0, -1)$ and $m_2 = (0, +1)$, as per the following **ansatz**, which was used as a basis of the **variational approximation** (but it does not produce an exact solution):

$$\phi_+ = A_1 \exp\left(-\alpha_1 r^2\right) - A_2 r \exp\left(-i\theta - \alpha_2 r^2\right),$$

$$\phi_- = A_1 \exp\left(-\alpha_1 r^2\right) + A_2 r \exp\left(+i\theta - \alpha_2 r^2\right).$$

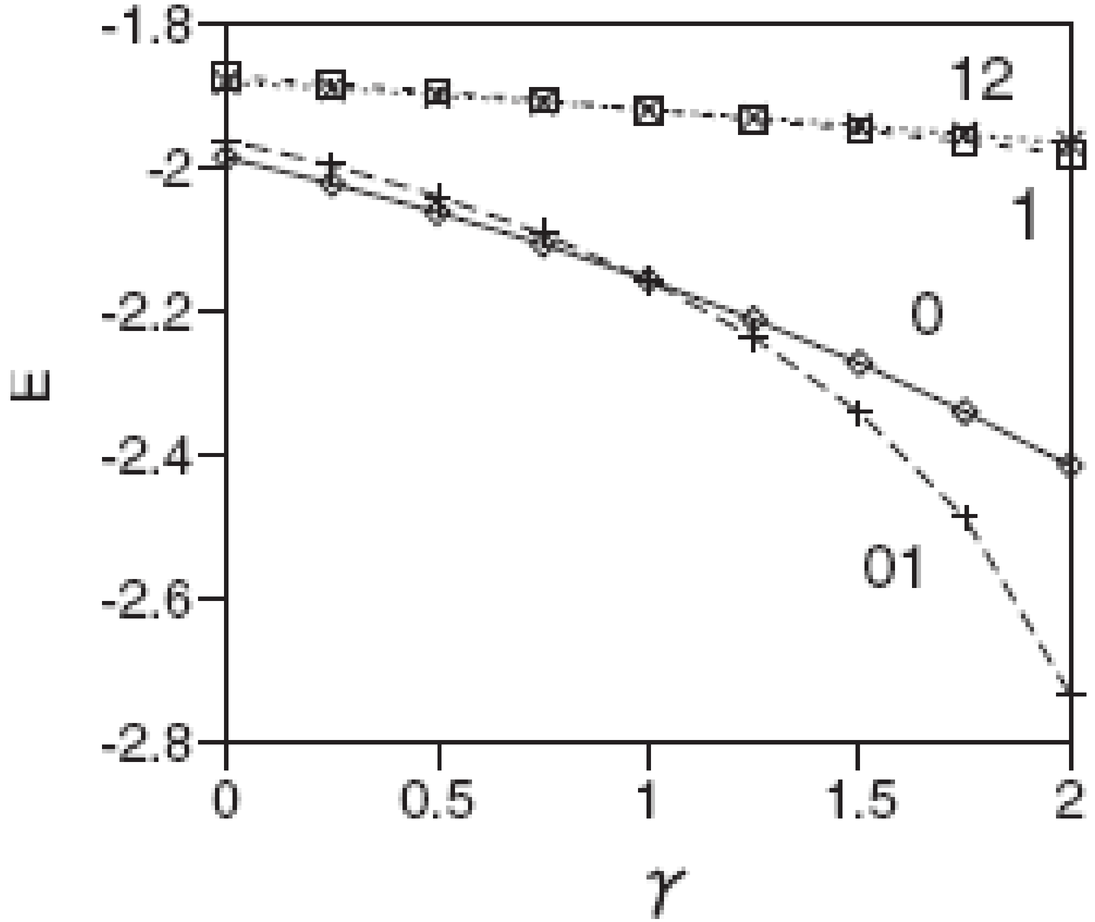
A typical example of the cross-section of the mixed mode:



The result is that the **mixed modes** are **unstable** at ***γ*** **< 1**, and **stable** at ***γ*** **> 1**, exactly when the semi-vortex is **stable** or **unstable**, respectively.
This **stability switch** between the **semi-vortex** and **mixed mode** with the increase of ***γ*** is explained by the fact that the **semi-vortex** and **mixed mode** are **ground states**, which realize the **minimum** of the system's energy, ***E***, precisely at ***γ*** **< 1** and ***γ*** **> 1**, respectively.

$$E = \iint \left\{ \frac{1}{2}(|\nabla\phi_+|^2 + |\nabla\phi_-|^2) - \frac{1}{2}(|\phi_+|^4 + |\phi_-|^4) - \gamma|\phi_+|^2|\phi_-|^2 + \frac{\lambda}{2}\left[\phi_+^*\left(\frac{\partial\phi_-}{\partial x} - i\frac{\partial\phi_-}{\partial y}\right) + \phi_-^*\left(-\frac{\partial\phi_+}{\partial x} - i\frac{\partial\phi_+}{\partial y}\right)\right] + \text{c.c.}\right\} dx dy,$$

The dependence of the energy of the **semi-vortex** ("**0**") and **mixed mode** ("**01**") on $\boldsymbol{\gamma}$, for a fixed value of the total norm, $\boldsymbol{N = 3.7}$:



(4f) The stabilization of 3D solitons by the SO coupling

PRL **115,** 253902 (2015) PHYSICAL REVIEW LETTERS week ending 18 DECEMBER 2015

Stable Solitons in Three Dimensional Free Space without the Ground State: Self-Trapped Bose-Einstein Condensates with Spin-Orbit Coupling

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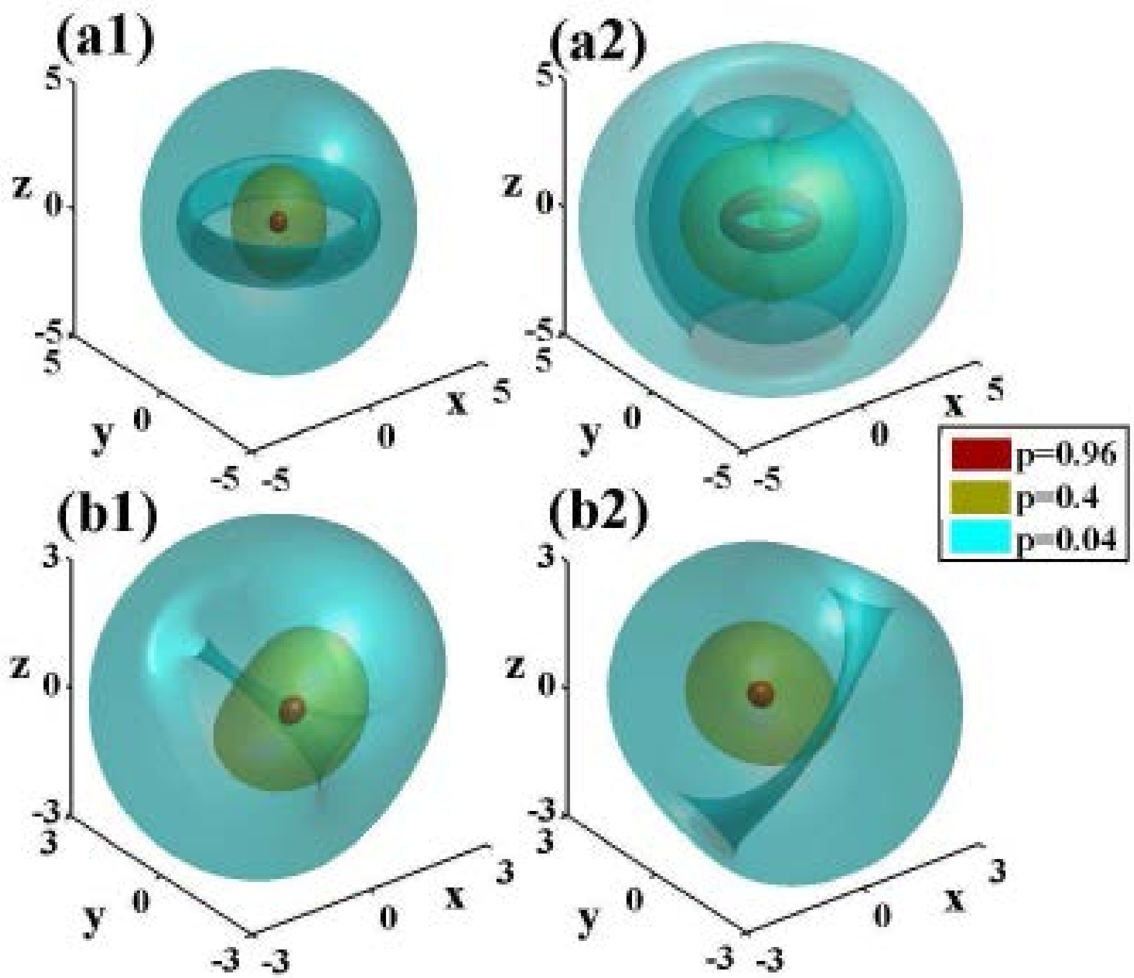
(Received 11 September 2015; published 17 December 2015)

The **3D** model with the **SO** coupling (in particular, of the ***Weyl type***) is based on the following system of **GP** equations for the two-component wave function, with the vector of **Pauli matrices**, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $g \equiv +1$, $\eta > 0$ (the ***attractive*** nonlinearity, which gives rise to the ***supercritical collapse*** in **3D**):

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 + i\lambda\nabla\cdot\boldsymbol{\sigma} \right.$$
$$\left. +g \begin{pmatrix} |\psi_+|^2 + \eta|\psi_-|^2 & 0 \\ 0 & |\psi_-|^2 + \eta|\psi_+|^2 \end{pmatrix} \right] \Psi = 0.$$

Examples of (***meta***) ***stable*** **3D** ***solitons***, with ***N* = 8** [(**a**), for ***η* = 0.3** – a semi-vortex;(**b**), for ***η* = 1.5** – a ***mixed mode***]:



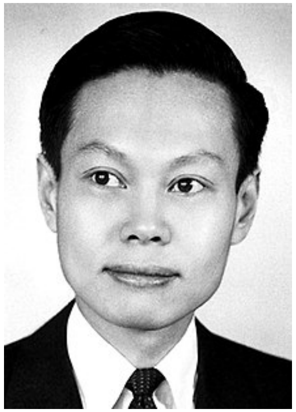
(5) The newest addition: stabilization of 3D and 2D "superfluid droplets" by quantum fluctuations

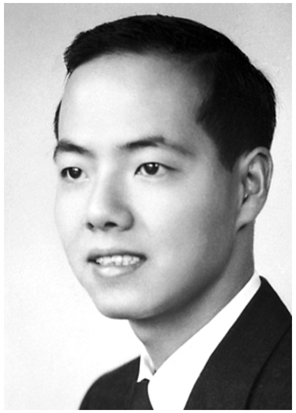
Starting in **2015**, a **stream** of theoretical predictions and ***direct experimental observations***, closely related to the topic of multidimensional solitons, has emerged: the creation of **3D** and **2D** soliton-like states in **binary BEC**, named "***quantum droplets***". They are made **stable** by the **Lee-Huang-Yang (LHY) correction** to the **BEC** dynamics, which represents effects of **quantum fluctuations (Bogoliubov's modes)** around the **mean-field** states**:**

T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties, Phys. Rev. **106**, 1135 **(1957)**.

Chen-Ning Yang in 1957 (currently **100** years old)

Tsung-Dao Lee in 1956 (currently **96** years old)





(5a) The idea of the **quantum droplets**, in **3D** and **2D** geometries, was proposed by **Dmitry Petrov** (Paris-Saclay):

PRL **115,** 155302 (2015) PHYSICAL REVIEW LETTERS week ending 9 OCTOBER 2015

Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture

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(Received 28 June 2015; published 7 October 2015)

PRL **117,** 100401 (2016) PHYSICAL REVIEW LETTERS week ending 2 SEPTEMBER 2016

Ultradilute Low-Dimensional Liquids

D. S. Petrov[1] and G. E. Astrakharchik[2]
[1]*LPTMS, CNRS, Univ. Paris Sud, Université Paris-Saclay, 91405 Orsay, France*
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(Received 24 May 2016; revised manuscript received 28 July 2016; published 1 September 2016)

The starting point is a system of coupled **GP** equations for two components of the mean-field wave function of a **binary BEC**, with **self-repulsion** in each component and **attraction** between the components, which **slightly exceeds** the self-repulsion.
Assuming ***equal wave functions*** of both components, $\boldsymbol{\Psi_1 = \Psi_2 \equiv \Psi}$, the system of two **GP** equations in **3D** may be reduced to the ***single*** one:

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\nabla^2\Psi - |\Psi|^2\,\Psi + \gamma\,|\Psi|^3\,\Psi,$$

where $\gamma > 0$ is an effective strength of the **quartic self - repulsion** term which represents the **LHY correction**.

(5b) Fundamental (zero-vorticity) droplets: solutions and experimental realizations

Stationary solutions for **quantum droplets** with chemical potential ***μ* < 0** are looked for as **Ψ** = exp(-*iμt*) **Φ**(*r*), with real functions **Φ**(*r*) satisfying equations

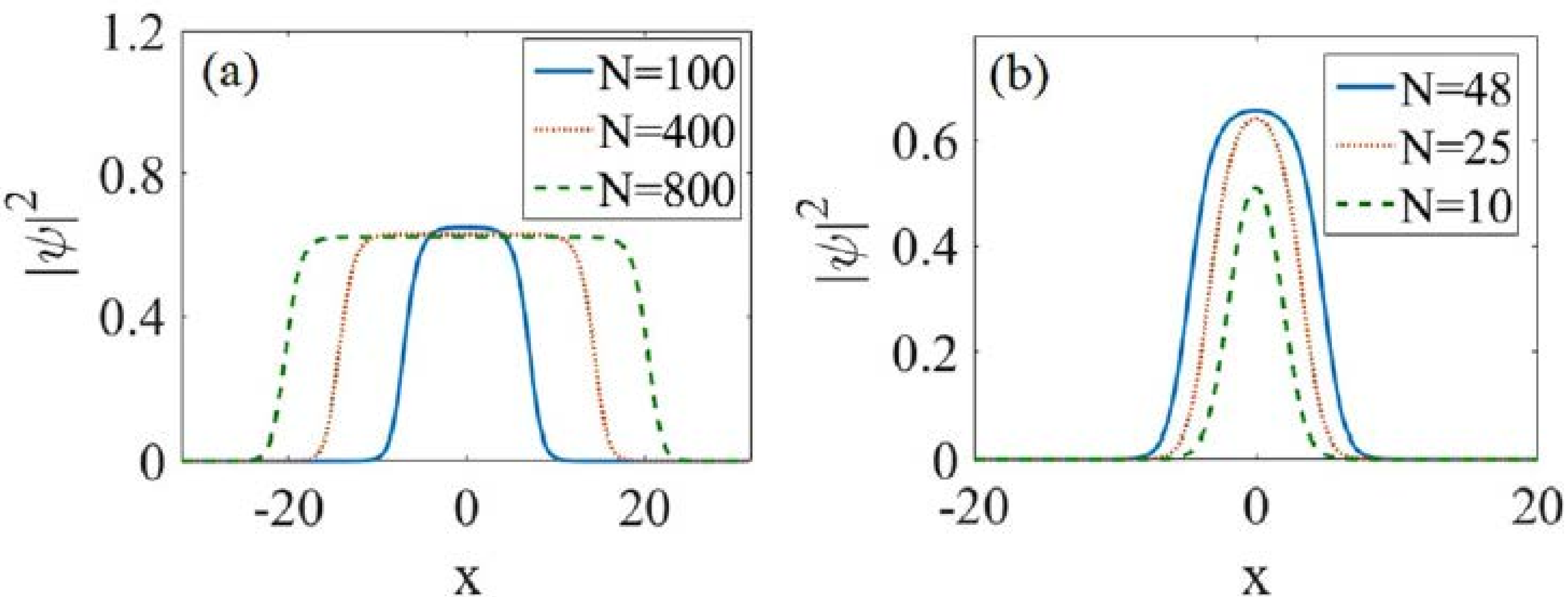
$$\mu\Phi = -\frac{1}{2}\left(\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr}\right) - \Phi^3 + \gamma\Phi^4.$$

Due to the **competition** between the **attractive** and **repulsive** nonlinearities, **density Φ^2** of the solutions cannot exceed the following **maximum value**:

$$\left(\Phi^2\right)_{\max} = \left(25/36\right)\gamma^{-2}.$$

For this reason, the quantum matter described by this equation may be considered as an effectively ***incompressible fluid***, hence the name of "***quantum droplets***".

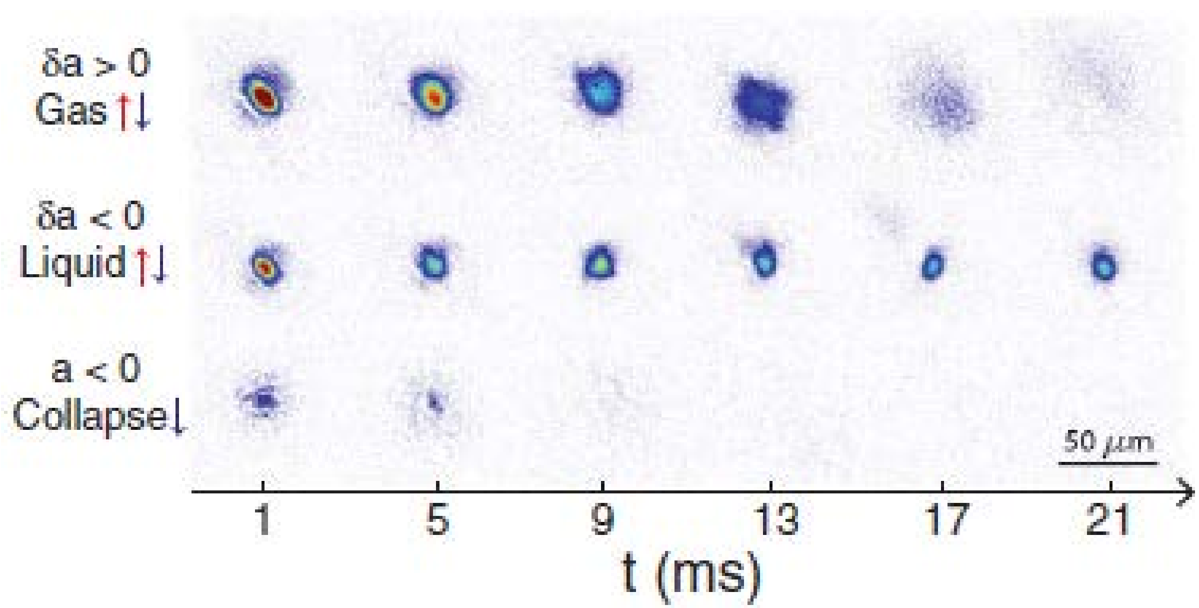
Cross sections of the **quantum-droplet** solutions with **small** (right, **bell-shaped**) and **large** (left, **flat-top** "droplets") total norms:



Experiment. The creation of **quasi-2D** (*oblate*) droplets, with ***aspect ratio* ~ 10:1** and lifetime **~10 ms**, built of **~10,000 ^{39}K atoms (in two different hyperfine states) per droplet**, was reported by the group of **Leticia Tarruell** from ICFO (Barcelona): **Science 359, 301 (2018)**

Quantum liquid droplets in a mixture of Bose-Einstein condensates

C. R. Cabrera,* L. Tanzi,* J. Sanz, B. Naylor, P. Thomas, P. Cheiney, L. Tarruell†



The creation of ***nearly isotropic*** **3D quantum droplets** (with negligible confinement in any direction) in the same atomic species, **^{39}K,** was reported by the group of **Massimo Inguscio** (Florence) :

PHYSICAL REVIEW LETTERS **120,** 235301 (2018)

Self-Bound Quantum Droplets of Atomic Mixtures in Free Space

G. Semeghini,[1,2,*] G. Ferioli,[1,2] L. Masi,[1,2] C. Mazzinghi,[1] L. Wolswijk,[1] F. Minardi,[2,1,3] M. Modugno,[4,5] G. Modugno,[1,2] M. Inguscio,[2,1] and M. Fattori[1,2]
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[2]*CNR Istituto Nazionale Ottica, 50019 Sesto Fiorentino, Italy*
[3]*Dipartimento di Fisica e Astronomia, Università di Bologna, 40127 Bologna, Italy*
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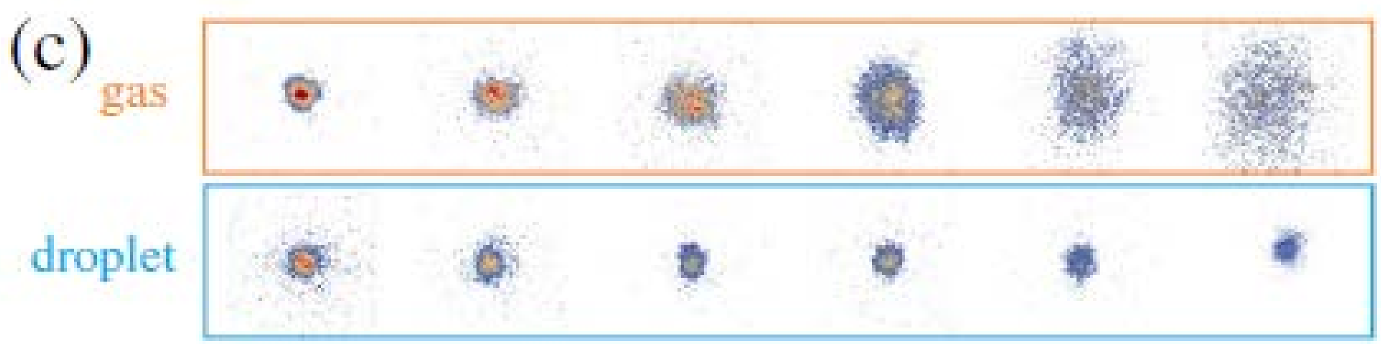
[5]*IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain*



(5c) Extension of the theory: quantum droplets with *embedded vorticity*

Vortex-droplet states, with **vorticity *S* = 1,2,3,*...***, were constructed in the **2D** model, looking for solutions (in the polar coordinates, ***r*** and ***θ***) as

$\Psi = \exp(-i\mu t + iS\theta)U(r)$, with $\mu < 0$.

PHYSICAL REVIEW A **98**, 063602 (2018)

Two-dimensional vortex quantum droplets

Yongyao Li,[1] Zhaopin Chen,[2] Zhihuan Luo,[3] Chunqing Huang,[1] Haishu Tan,[1] Wei Pang,[4,*] and Boris A. Malomed[2,1]
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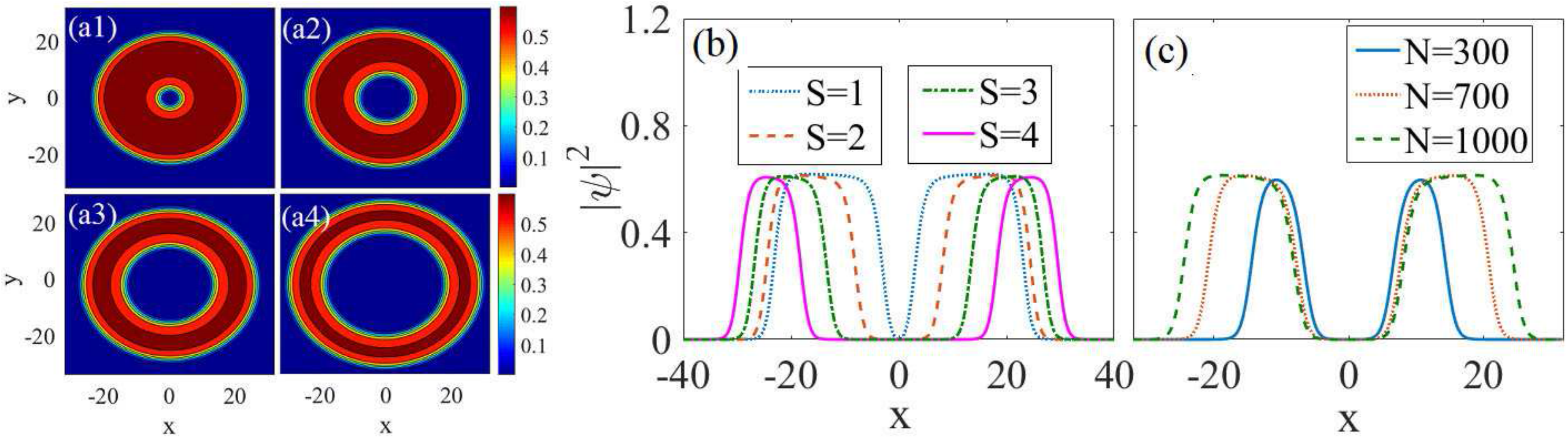
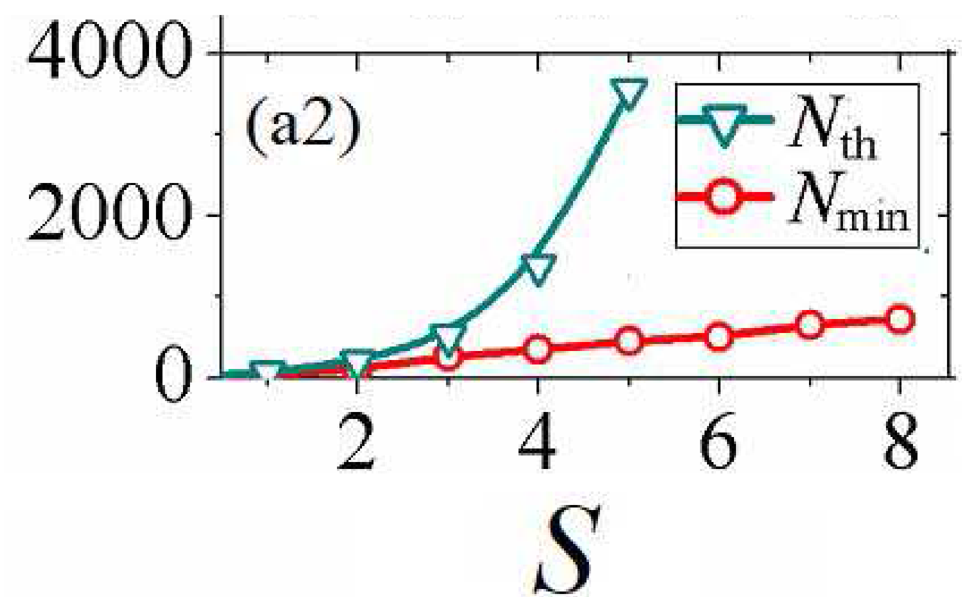
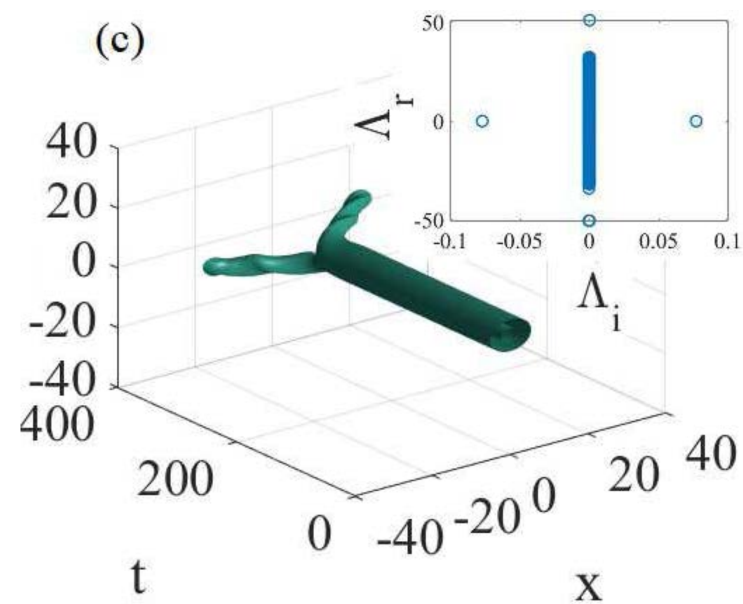
[2]*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
[3]*College of Electronic Engineering, South China Agricultural University, Guangzhou 510642, China*
[4]*Department of Experiment Teaching, Guangdong University of Technology, Guangzhou 510006, China*


(a1)
(a2)
(a3)
(a4)
y
x
(b)
S=1
S=2
S=3
S=4
$|\psi|^2$
x
(c)
N=300
N=700
N=1000
x


(a2)
N_{th}
N_{min}
S


(c)
Λ_r
Λ_i
t
x

A more advanced theoretical result: prediction of **stability regions** for **fully 3D** droplets, with embedded vorticities ***S* = 1** and **2**:

PHYSICAL REVIEW A **98**, 013612 (2018)

Three-dimensional droplets of swirling superfluids

Yaroslav V. Kartashov,[1,2,*] Boris A. Malomed,[3] Leticia Tarruell,[1] and Lluis Torner[1,4]

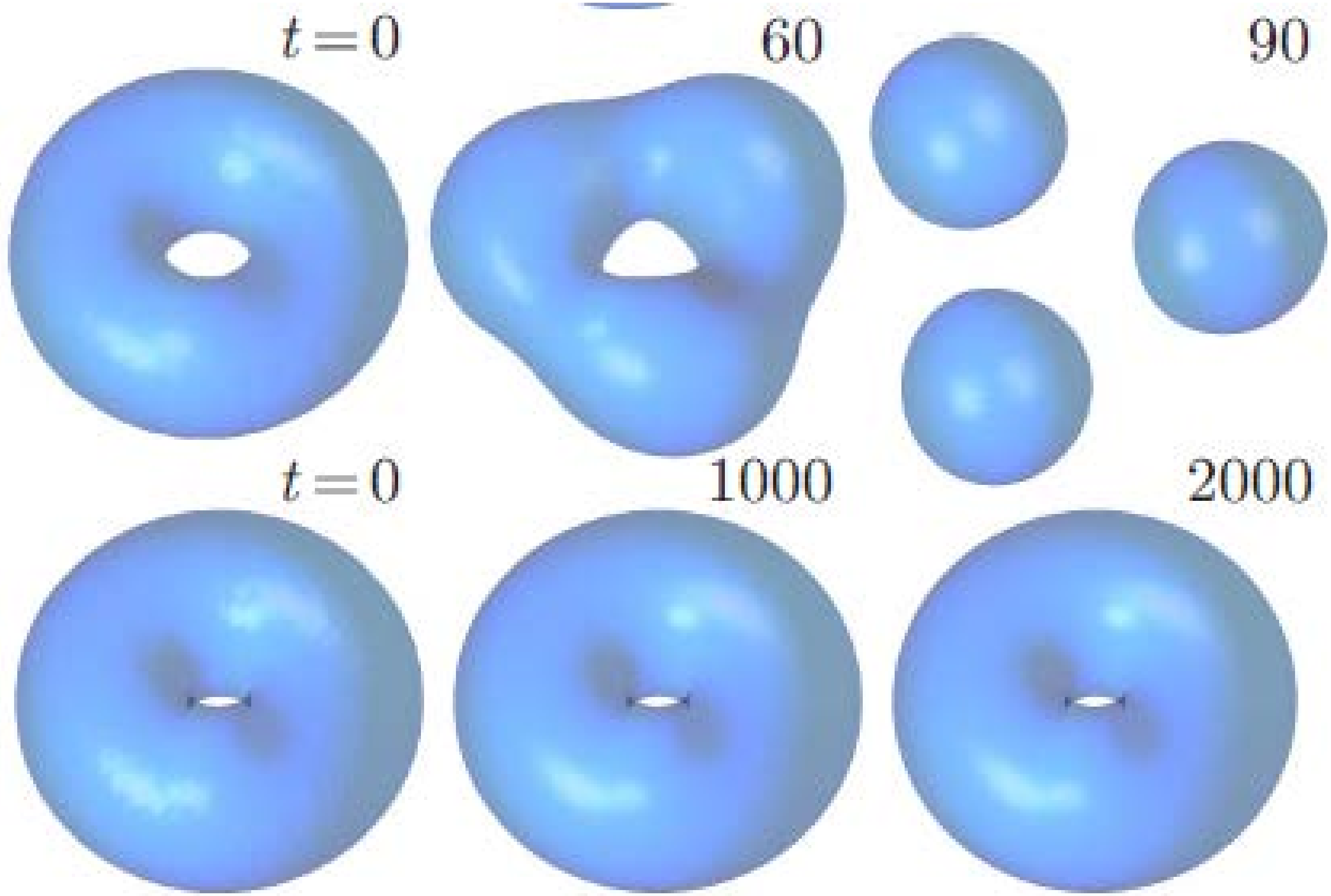
[1]*ICFO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels, Barcelona, Spain*

[2]*Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow, 108840, Russia*

Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, and Center for Light-Matter Interaction, Tel Aviv University, 69978 Tel Aviv, Israel and ITMO University, St. Petersburg 197101, Russia

[4]*Universitat Politecnica de Catalunya, 08034, Barcelona, Spain*

Examples of ***splitting*** of an ***unstable*** **3D** ***vortex droplet*** (the **top row**), and of the evolution of a ***stable*** one (the **bottom row**). In both cases, ***S*** **= 1**.



A relatively recent review on the topic of **quantum droplets** (theory and experiment):

Frontiers of Physics
https://doi.org/10.1007/s11467-020-1020-2
Front. Phys. 16(3), 32201 (2021)

REVIEW ARTICLE

A new form of liquid matter: Quantum droplets

Zhi-Huan Luo[1], Wei Pang[2], Bin Liu[3], Yong-Yao Li[3,†], Boris A. Malomed[4,5]

(6) Conclusions

Recent theoretical and experimental studies have led to the prediction and, in some cases, experimental creation of ***stable*** self-trapped modes in the form of **fundamental** and **vortex solitons** in **2D** and **3D** geometries. Especially interesting are the predicted possibilities for the creation of (meta)stable **semi-vortices** by means of the **SO (spin-orbit) coupling** for the binary **BEC** with cubic attractive interactions (something which was previously considered ***absolutely impossible***), as well as the

theoretically predicted and ***experimentally realized*** creation of **3D superfluid droplets**, stabilized by **quantum fluctuations**.

Generally, the theoretical studies of multidimensional solitons have advanced much farther than the experimental work. Creation of stable **2D** and **3D** solitons in ***real experiments*** remains a challenging objective.

Thank you for your interest!
感谢您的关注!

Copies of this presentation, and/or of articles mentioned in it, can be requested from

malomed@tauex.tau.ac.il