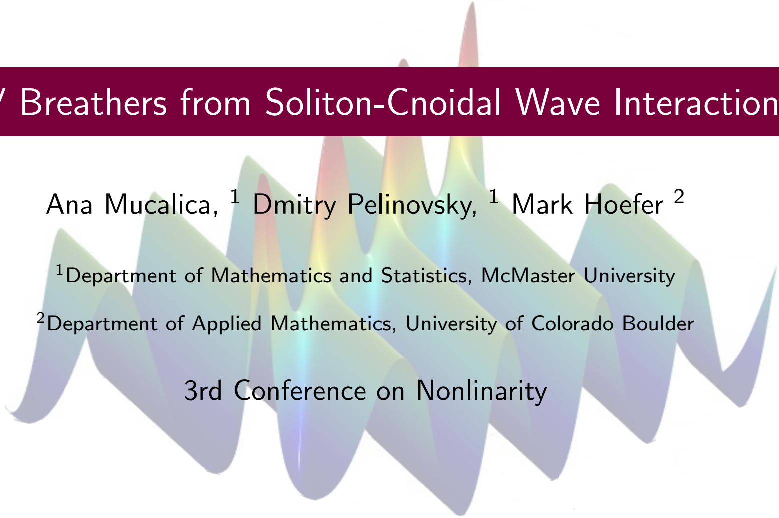
KdV Breathers from Soliton-Cnoidal Wave Interactions



Ana Mucalica, [1] Dmitry Pelinovsky, [1] Mark Hoefer [2]

[1]Department of Mathematics and Statistics, McMaster University

[2]Department of Applied Mathematics, University of Colorado Boulder

3rd Conference on Nonlinarity

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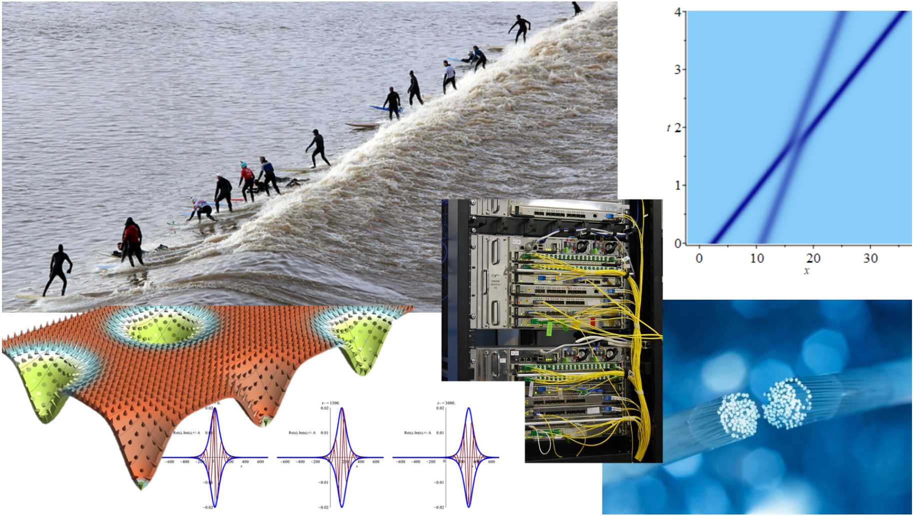
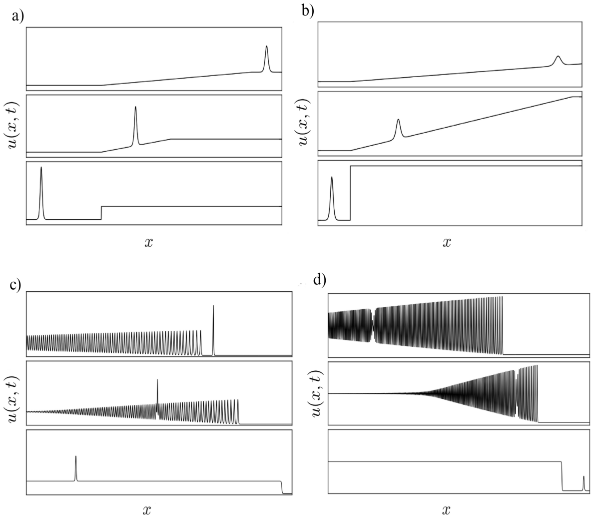
Solitons appear in nature under number of circumstances...



Motivation for Soliton-Dispersive Wave Interactions

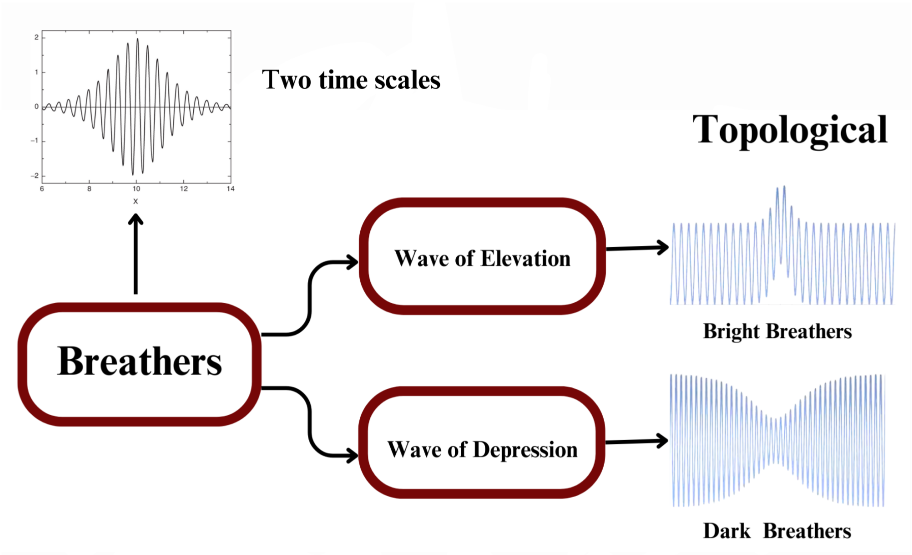


a) Soliton–RW tunneling.
b) Soliton–RW trapping.

c) Soliton–DSW tunneling.
d) Soliton–DSW trapping.

M. J. Ablowitz, J. T. Cole, M. A. Hoefer, et al. ArXiv 2211.14884v1, (2022).

Breather Classification



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Main Model

We are dealing with the canonical model for the shallow water waves, the Korteweg–de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where t is the time evolution, x is the spatial coordinate for the wave propagation, and u is the fluid velocity.

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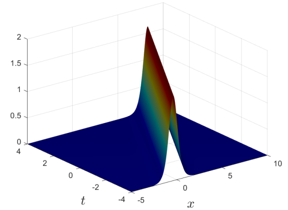
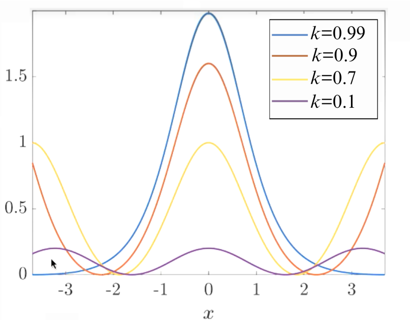
One-soliton solution of the KdV equation (1)

$$u(x,t) = 2\mu^2 \operatorname{sech}^2\left(\mu(x - 4\mu^2 t - x_0)\right)$$



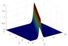
KdV equation (1) has a family of traveling periodic wave solutions

$$u(x,t) = 2k^2 \mathrm{cn}^2(x - ct; k), \qquad c = 4(2k^2 - 1).$$



Idea: Use the one-fold DT to superimpose one soliton on a cnoidal wave background.

Tools Used for Construction of Interaction Solutions



Inverse Scattering Transform (IST):

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u \tag{2}$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}, \tag{3}$$

λ is the time-independent spectral parameter.

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⇨ (3) represents the time evolution of the eigenfunctions

- Darboux Transformation:

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

u is the known solution for the KdV equation (1).

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Lamé equation as the Spectral Problem

The spectral problem (2) with the normalized cnoidal wave potential is known as the Lamé equation

$$v''(x) - 2k^2 \mathrm{sn}^2(x,k) v(x) + \eta v(x) = 0, \quad \eta := \lambda + 2k^2 \tag{4}$$

where the single variable x stands for $x - c_0 t$.

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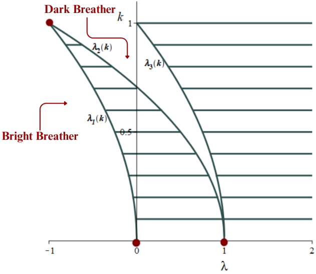


Figure: Floquet spectrum of the Lamé equation (4) with the band edges $\lambda_{1,2,3}(k)$ corresponding to three particular solutions $v_{1,2,3}(x)$.

The Eigenfunctions

Two linearly independent solutions of the Lamé equation (4) for $\lambda \neq \lambda_{1,2,3}(k)$ are given by the functions

$$v_{\pm}(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)}, \tag{5}$$

where $\alpha \in \mathbb{C}$ is found from $\lambda \in \mathbb{R}$ by using the characteristic equation $\eta = k^2 + \mathrm{dn}^2(\alpha, k)$ and the Jacobi zeta function is $Z(\alpha) := \frac{\Theta'(\alpha)}{\Theta(\alpha)}$.

$$H(x) = \theta_1\left(\frac{\pi x}{2K(k)}\right), \quad \theta_1(u) = 2\sum_{n=1}^{\infty}(-1)^{n-1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u$$

$$\Theta(x) = \theta_4\left(\frac{\pi x}{2K(k)}\right), \quad \theta_4(u) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nu$$

Bright Breather on the Cnoidal Wave Background

Theorem

There exists an exact solution to the KdV equation (1) in the form

$$u(x,t) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau(x,t), \tag{6}$$

where the τ*-function is given by*

$$\tau(x,t) := \Theta(x - c_0 t + \alpha_b)e^{\kappa_b(x - c_b t + x_0)} + \Theta(x - c_0 t - \alpha_b)e^{-\kappa_b(x - c_b t + x_0)}, \tag{7}$$

where $x_0 \in \mathbb{R}$ *is arbitrary and* $\alpha_b \in (0, K(k))$, $\kappa_b > 0$, *and* $c_b > c_0$ *are uniquely defined from* $\lambda \in (-\infty, \lambda_1(k))$.

Solution Surface for Bright Breather on Cnoidal Wave Background

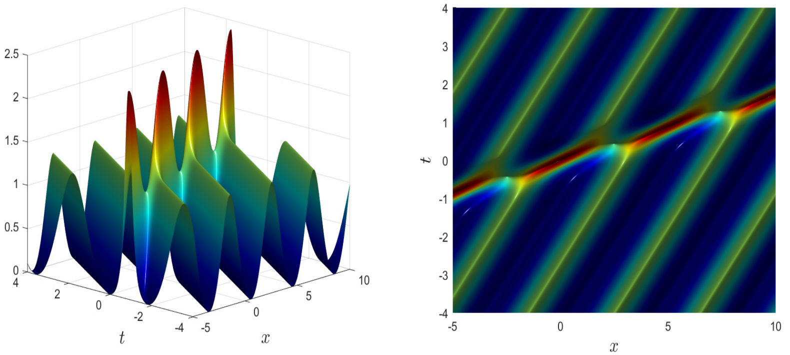


Figure: Bright breather on the cnoidal wave with $k = 0.8$ for $\lambda = -1.2$ and $x_0 = 0$.

Dark Breather on the Cnoidal Wave Background

Theorem

There exists an exact solution to the KdV equation (1) in the form

$$u(x,t) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau(x,t), \tag{8}$$

where the τ-function is given by

$$\tau(x,t) := \Theta(x - c_0 t + \alpha_d) e^{-\kappa_d(x - c_d t + x_0)} + \Theta(x - c_0 t - \alpha_d) e^{\kappa_d(x - c_d t + x_0)}, \tag{9}$$

where $x_0 \in \mathbb{R}$ is arbitrary and $\alpha_d \in (0, K(k))$, $\kappa_d > 0$, and $c_d < c_0$ are uniquely defined from $\lambda \in (\lambda_2(k), \lambda_3(k))$.

Solution Surface for Dark Breather on Cnoidal Wave Background

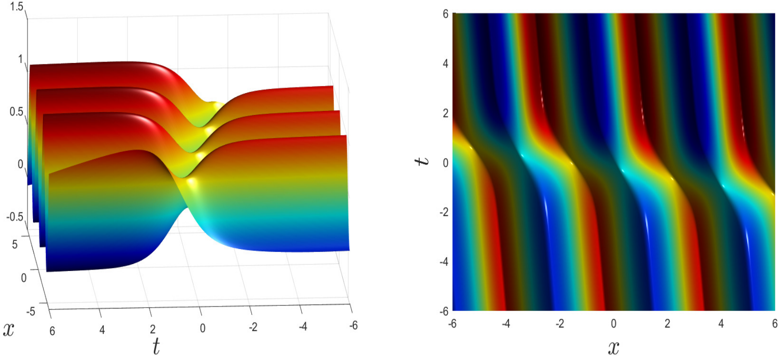
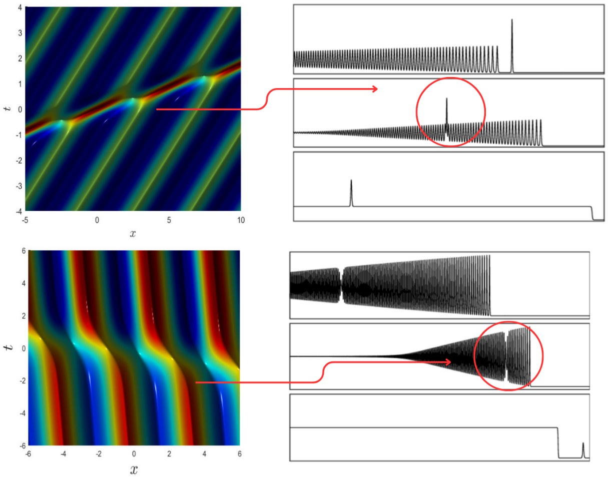


Figure: Dark breather on the cnoidal wave background with $k = 0.7$ for $\lambda = 0.265$ and $x_0 = 0$.

Direction of Breather Propagation



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Experimental Confirmation of the Breather Existence

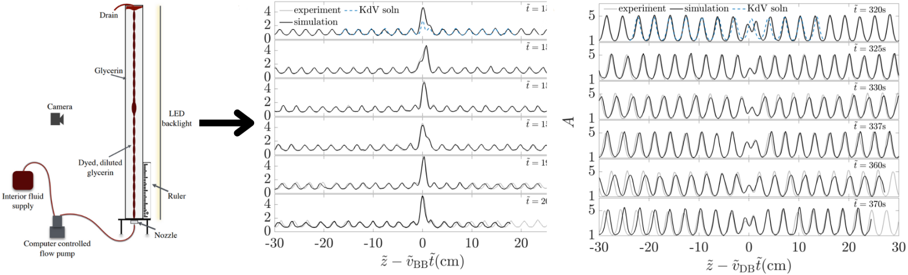


Figure: Experiment (light gray) compared with simulation of the conduit equation (black) with initial conditions from experiment, and the KdV breather solution (blue). Left: Bright Breather. Right: Dark Breather.

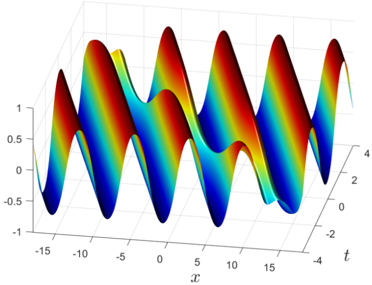
Y. Mao, M. A. Hoefer, et al. ArXiv: 2302.11161, (2023).

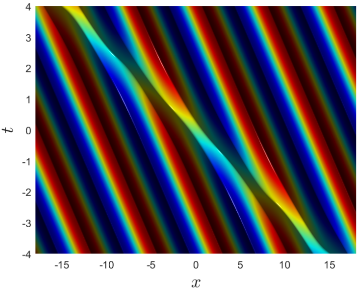
Future Work

Defocusing modified Korteweg–de Vries (mKdV) equation:

$$u_t - 6u^2 u_x + u_{xxx} = 0, \qquad (10)$$

$$u(x,t) = k\,\mathrm{sn}(x + c_0 t; k), \qquad c_0 = 1 + k^2.$$





Published Work

M. Hoefer, A. Mucalica and D.E. Pelinovsky, "KdV breathers on a cnoidal wave background", J. Phys. A: Math. Theor. **56** 185701 (2023). DOI: 10.1088/1751-8121/acc6a8

A. Mucalica and D.E. Pelinovsky, "Solitons on the rarefaction wave background via the Darboux transformation", Proc. R. Soc. A **478** (2022). DOI:10.1098/rspa.2022.0474

Thank you!