Realization of the brachistochronic motion of a rigid body in a vertical plane using real rough centroids

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- **The rigid-body motion in a minimum time, between two specified positions in a vertical plane, is considered for the specified value of the initial mechanical energy.**
- **The problem is formulated and solved in a closed form, which is a contribution of this paper, considering non-linear differential equations of the two-point boundary value problem of Pontryagin's maximum principle.**
- **It is shown that the solution thus obtained also represents global minimum time for motion.**
- **In the spirit of the classical brachistochrone problem of the particle the realization of this motion is also achieved exclusively by ideal mechanical constraints (by centroides).**
- **The laws of change of the tangential and normal component of the constraint reaction are obtained in the analytical form. Based on these laws, the dependence of the coefficient of sliding friction on time is obtained.**
- **Maximum value of this coefficient must be smaller than the Coulomb coefficient of friction for the case of real rough surface. If this is not satisfied, an appropriate optimal control problem is formulated.**

- **Classical Bernoulli's problem of determining the brachistochrone for the particle in a vertical plane:**

Bernoulli, J.: *Problema novum ad cuius solutionem Mathematici invitantur (A new problem that mathematicians are invited to solve).* Acta Eruditorum. 15 (1696), 264–269.

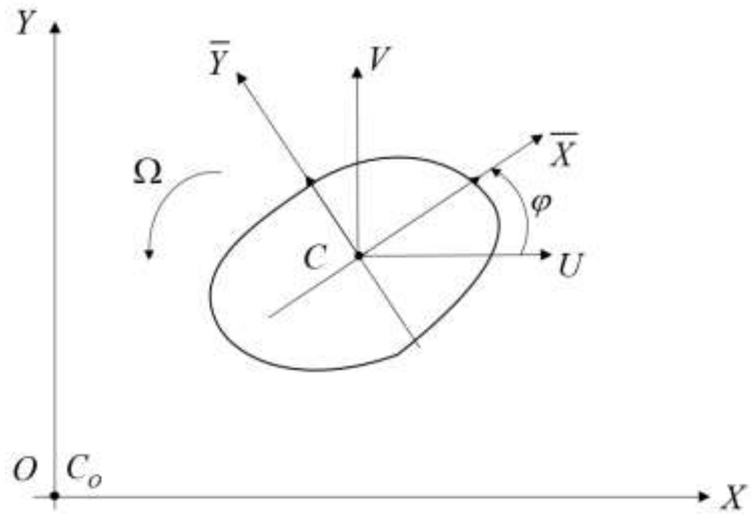
- **A more detailed review of literature devoted to these generalizations can be found in a doctoral dissertation [2] (Advisor: Oleg Yu. Cherkasov):**

Zarodnyuk, A.V., *Optimization of controlled descent and generalized brachistochrone problems (in Russian)*, PhD, Moscow State University, Faculty of Mechanics and Mathematics, 2018.

- **Our cooperation (unilateral nonholonomic constraint):**

Obradović, A., Cherkasov, O.Y., Miličić, L., The Brachistochronic Motion of Chaplygin Sleigh in a Vertical Plane with Unilateral Nonholonomic Constraint, Proceedings of 9th International Congress of Serbian Society of Mechanics Vrnjačka Banja, Serbia, July 5-7, 2023

Rigid body (Fig. 1.) of mass m and the radius of inertia i is moving in a vertical plane. The body position is specified at the initial and final moment.
Initial value of mechanical energy: mgL .



During brachistochronic motion, the mechanical energy remains unchanged:

$$\frac{1}{2}m\left(U^2+V^2+i^2\Omega^2\right)+mgY=mgL \tag{1}$$

Kinematic differential equations:

$$\begin{aligned}\dot{X} &= U\\ \dot{Y} &= V\\ \dot{\varphi} &= \Omega\end{aligned} \tag{2}$$

Dimensionless quantities:

$$X = ix,\quad Y = iy,\quad \Omega = \omega\sqrt{\frac{g}{i}},\quad V = v\sqrt{gi},\quad U = u\sqrt{gi},\quad t = \tau\sqrt{\frac{i}{g}},\quad L = il,$$
$$F_N = mgF_n,\qquad F_T = mgF_t,\quad \bar{X} = i\bar{x},\quad \bar{Y} = i\bar{y} \tag{3}$$

Equations of state:

$$\begin{aligned}x' &= u\\ y' &= v\\ \varphi' &= \omega\end{aligned} \tag{4}$$

The notation $(\ldots)'$ represents differentiation with respect to dimensionless time.

Conservation of mechanical energy:

$$\boldsymbol{u^2 + v^2 + \omega^2 + 2y - 2l = 0} \tag{5}$$

Initial conditions of motion:

$$\tau_0 = 0 \quad x(\tau_0) = 0 \quad y(\tau_0) = 0 \quad \varphi(\tau_0) = 0 \tag{6}$$

Final position:

$$\tau_1 = ? \quad x(\tau_1) = x_1 \quad y(\tau_1) = y_1 \quad \varphi(\tau_1) = \varphi_1 \tag{7}$$

Brachistochronic motion consists of determining the optimal controls:

$$u = u(\tau) \quad v = v(\tau) \quad \omega = \omega(\tau) \tag{8}$$

Functional being minimized in this problem:

$$J = \int_0^{\tau_1} d\tau = \tau_1 \tag{9}$$

Pontryagin's function for the case of time minimization:

$$\boldsymbol{H = -1 + \lambda_x u + \lambda_y v + \lambda_\varphi \omega - \mu(u^2 + v^2 + \omega^2 + 2y - 2l)} \tag{10}$$

μ is the multiplier corresponding to the constraint of mechanical energy (5) and $\boldsymbol{\lambda_x}$, $\boldsymbol{\lambda_y}$, $\boldsymbol{\lambda_\varphi}$ are the co-state variables.

The co-state system of differential equations:

$$\boldsymbol{\lambda_x{}' = 0 \quad \lambda_y{}' = 2\mu \quad \lambda_\varphi{}' = 0} \tag{11}$$

Optimality conditions:

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial v} = 0, \quad \frac{\partial H}{\partial \omega} = 0 \tag{12}$$

Optimal controls:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{\frac{\lambda_x}{2\mu}} \\ \boldsymbol{v} &= \boldsymbol{\frac{\lambda_y}{2\mu}} \\ \boldsymbol{\omega} &= \boldsymbol{\frac{\lambda_\varphi}{2\mu}} \end{aligned} \tag{13}$$

Multiplier $\boldsymbol{\mu}$, is defined from conditions:

$$H(\tau)=0 \tag{14}$$

$$\boldsymbol{\mu}(\boldsymbol{\tau}) = \frac{\mathbf{1}}{\mathbf{4}(\boldsymbol{l}-\boldsymbol{y}(\boldsymbol{\tau}))} > \mathbf{0} \tag{15}$$

Second-order conditions:

$$\frac{\partial^2 \boldsymbol{H}}{\partial \boldsymbol{u}^2} = -\mathbf{2}\boldsymbol{\mu} < \mathbf{0}, \quad \frac{\partial^2 \boldsymbol{H}}{\partial \boldsymbol{v}^2} = -\mathbf{2}\boldsymbol{\mu} < \mathbf{0}, \quad \frac{\partial^2 \boldsymbol{H}}{\partial \boldsymbol{\omega}^2} = -\mathbf{2}\boldsymbol{\mu} < \mathbf{0} \tag{16}$$

Optimal controls:

$$\begin{aligned} \boldsymbol{u} &= \mathbf{2}(\boldsymbol{l}-\boldsymbol{y})\boldsymbol{\lambda}_{\boldsymbol{x}} \\ \boldsymbol{v} &= \mathbf{2}(\boldsymbol{l}-\boldsymbol{y})\boldsymbol{\lambda}_{\boldsymbol{y}} \\ \boldsymbol{\omega} &= \mathbf{2}(\boldsymbol{l}-\boldsymbol{y})\boldsymbol{\lambda}_{\boldsymbol{\varphi}} \end{aligned} \tag{17}$$

Differential equations of TPBVP:

$$
\begin{aligned}
&x' = 2(l-y)\lambda_x && \lambda_x{}' = 0 \\
&y' = 2(l-y)\lambda_y && \lambda_y{}' = \frac{1}{2(l-y)} \\
&\varphi' = 2(l-y)\lambda_\varphi && \lambda_\varphi{}' = 0
\end{aligned}
\tag{18}
$$

General solutions in the analytical form:

$$
\begin{aligned}
\boldsymbol{y} &= \boldsymbol{l} - \frac{\boldsymbol{1+cos(p\tau+\alpha)}}{\boldsymbol{p^2}} \\
\boldsymbol{x} &= \frac{\boldsymbol{2\lambda_x}}{\boldsymbol{p^2}}\left(\boldsymbol{\tau} + \frac{\boldsymbol{1}}{\boldsymbol{p}}\boldsymbol{sin(p\tau+\alpha)}\right) + \boldsymbol{C_1} \\
\boldsymbol{\varphi} &= \frac{\boldsymbol{2\lambda_\varphi}}{\boldsymbol{p^2}}\left(\boldsymbol{\tau} + \frac{\boldsymbol{1}}{\boldsymbol{p}}\boldsymbol{sin(p\tau+\alpha)}\right) + \boldsymbol{C_2}
\end{aligned}
\tag{19}
$$

where $(p, \alpha, \lambda_x, \lambda_\varphi, C_1, C_2)$ are determined together with unknown moment τ_1 from (5), (6) and (7).

Non-linear ordinary equations, whose solution gives values p, α, τ_1:

$$0 = l - \frac{1 + cos(\alpha)}{p^2}$$

$$y_1 = l - \frac{1 + cos(p\tau_1 + \alpha)}{p^2}$$

$$2\left(\frac{x_1 p^2}{2\left(\tau_1 + \frac{1}{p} sin(p\tau_1 + \alpha) - \frac{1}{p} sin(\alpha)\right)}\right)^2 + \left(\frac{p\ sin(p\tau_1 + \alpha)}{2(1 + cos(p\tau_1 + \alpha)}\right)^2 = \frac{p^2}{2(1 + cos(p\tau_1 + \alpha)} \quad (20)$$

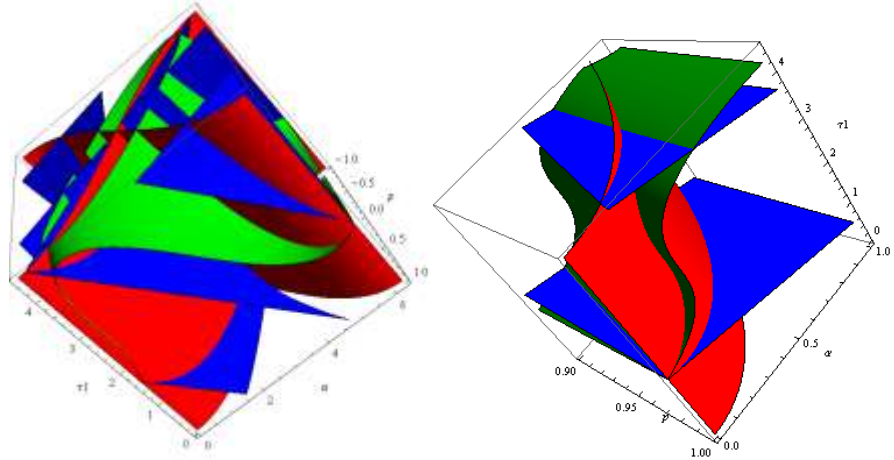
Global minimum time τ_1 should be sought among their multiple solutions.
Limits of all possible solutions:

$$0 \le \alpha < 2\pi, \qquad -\sqrt{\frac{2}{l}} \le p \le \sqrt{\frac{2}{l}}, \qquad 0 \le \tau_1 \quad (21)$$

Parameters of the task:

$$\boldsymbol{l = 2,\ x_1 = -\varphi_1 = \frac{\pi+2}{2\sqrt{2}},\ y_1 = 1} \tag{22}$$

Three surfaces of different colors, each of which corresponds to the fulfillment of one of the equations (20).



Analytical solutions of the corresponding TPBVP represents global minimum time:

$$\boldsymbol{p = \pm 1,\ \tau_1 = \frac{\pi}{2},\ \alpha = 0,}$$

$$\boldsymbol{x = \frac{(\tau+sin\tau)}{\sqrt{2}},\quad y = 1 - cos\tau,\quad \varphi = -\frac{(\tau+sin\tau)}{\sqrt{2}}} \tag{23}$$

$$\boldsymbol{\lambda_x = \frac{1}{2\sqrt{2}},\quad \lambda_y = \frac{sin\tau}{2(1+cos\tau)},\quad \lambda_\varphi = -\frac{1}{2\sqrt{2}},}$$

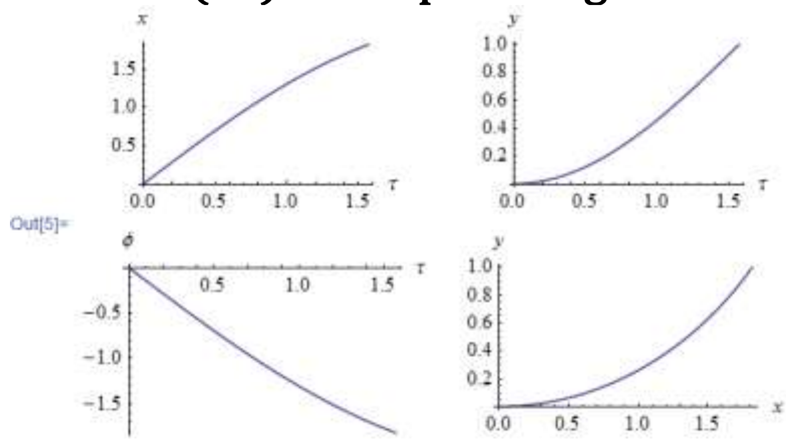
First two following solutions (numerical). Both variants lead to identical final equations of motion:

$$\begin{aligned} p &= \;\;\, 0.954626,\ \tau_1 = 4.20981,\ \alpha = 0.604787, \\ p &= -0.954626,\ \tau_1 = 4.20981,\ \alpha = 5.678340, \end{aligned} \tag{24}$$

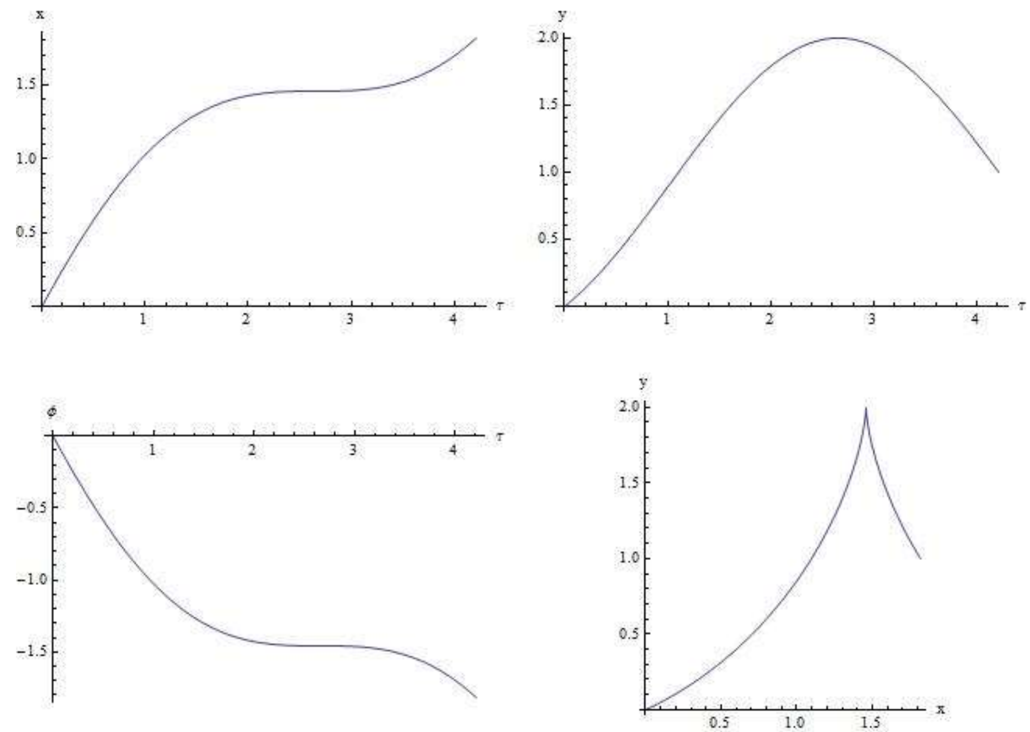
The possibility of some other solutions is not excluded however they would correspond to larger values of τ_1 and are not of interest to this problem.

It can be also noticed that the mass center trajectory is a deformed cycloid with the coefficient $\frac{1}{\sqrt{2}}$.

Analytical solutions (23) correspond to global time minimum:



Local time minimum (24):



Realization of the brachistochronic - rolling without slip of a moving centroid on a fixed one.

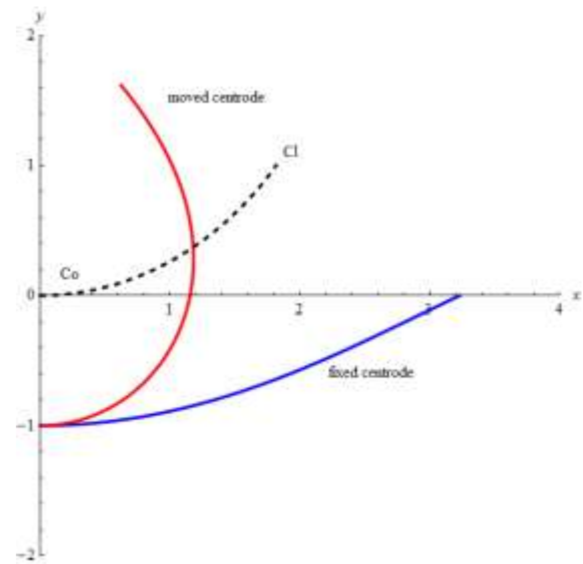
Parametric equations of a fixed centroid:

$$x_P = x - \frac{y'}{\varphi'} \quad , \quad y_P = y + \frac{x'}{\varphi'} \tag{25}$$

and moving centroid (in the moving coordinate system):

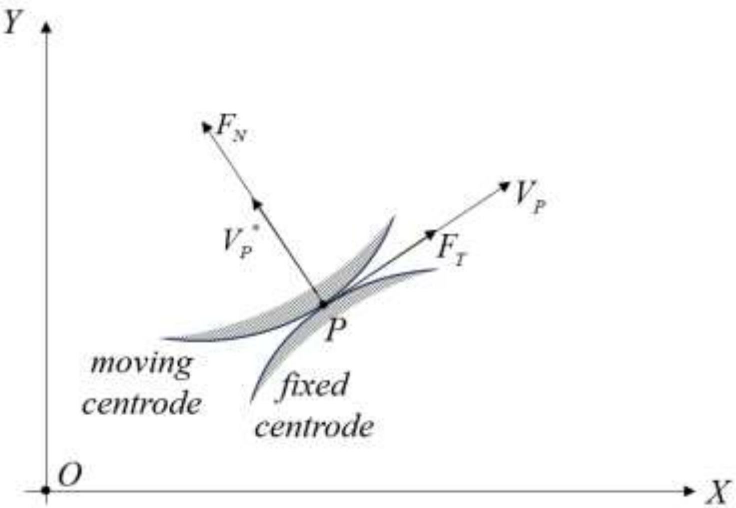
$$\overline{x_P} = \frac{(x' sin\varphi - y' cos\varphi)}{\varphi'} \quad , \quad \overline{y_P} = \frac{(y' sin\varphi + x' cos\varphi)}{\varphi'} \tag{26}$$

Mass center trajectory, fixed centroid and moving centroid at the initial position:



Tangential and normal components of the constraint reaction must satisfy the Coulomb friction laws during motion (μ_0 - Coulomb coefficient of friction):

$$-\boldsymbol{\mu_0} \leq \boldsymbol{\mu} = \frac{\boldsymbol{F_t}}{\boldsymbol{F_n}} \leq \boldsymbol{\mu_0} \tag{27}$$



Dimensionless velocity $\mathbf{v_P}$ of contact point P and vector $\mathbf{v_P^*}$ normal to it:

$$\boldsymbol{v_{Px}} = \boldsymbol{x'_P} \quad , \quad \boldsymbol{v_{Py}} = \boldsymbol{y'_P} \quad , \quad \boldsymbol{v^*_{Px}} = -\boldsymbol{y'_P} \quad , \quad \boldsymbol{v^*_{Py}} = \boldsymbol{x'_P} \tag{28}$$

The law of motion of the center of mass:

$$u' = F_{nx} + F_{tx} \ , \quad v' = F_{ny} + F_{ty} - 1 \tag{29}$$

Components of the constraint reaction at point P:

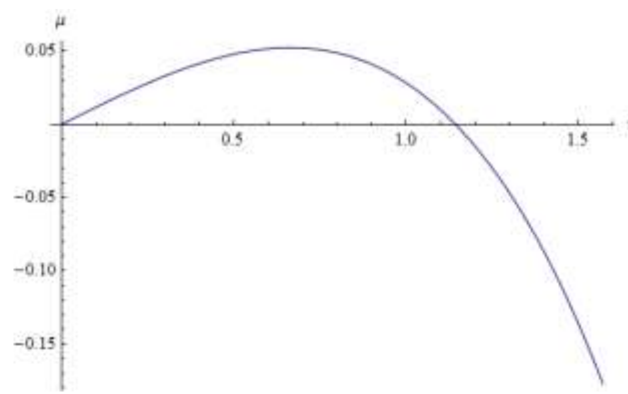
$$F_t = \frac{u' v_{Px} + (1+v')\, v_{Py}}{v_P} \qquad F_n = \frac{u' v^*_{Px} + (1+v') v^*_{Py}}{v_P} \tag{30}$$

Coefficient of friction:

$$\mu = \frac{u' v_{Px} + (1+v')\, v_{Py}}{u' v^*_{Px} + (1+v') v^*_{Py}} \tag{31}$$

which, can be obtained in the analytical form:

$$\mu = \frac{(-1+4\cos\tau+\cos 2\tau)\,\mathrm{tg}\frac{\tau}{2}}{4\sqrt{2}(2+\cos\tau)} \tag{32}$$



Tangential component of the constraint reaction changes direction by $\boldsymbol{\tau} = \mathbf{1.14372}$
By absolute value, the highest necessary Coulomb coefficient is at the end of motion and amounts to $\boldsymbol{\mu}^* = \mathbf{0.176777}$.

Such method of realizing is possible if the surfaces are real rough surfaces with the coefficient $\boldsymbol{\mu}_0 > \boldsymbol{\mu}^*$.

If the Coulomb coefficient of sliding friction were lower than mentioned limit, the problem of brachistochronic motion would have to be formulated in a more complex form.

In that case, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\omega}$ would become the quantities of state, relation (5) would join the initial conditions (6):

$$\boldsymbol{u}(\mathbf{0})^2 + \boldsymbol{v}(\mathbf{0})^2 + \boldsymbol{\omega}(\mathbf{0})^2 + \mathbf{2}\boldsymbol{y}(\mathbf{0}) - \mathbf{2}\boldsymbol{l} = \mathbf{0} \qquad (33)$$

In the equations of state, apart from kinematic equations (4), dynamic equations would have to be incorporated too (the controls would be $\mathbf{u_x}, \mathbf{u_y}, \mathbf{u_\varphi}$) :

$$\begin{aligned} u' &= u_x \\ v' &= u_y \\ \omega' &= u_\varphi \end{aligned} \tag{34}$$

Based on relation (5), the restriction is imposed on control in the form of equality:

$$u u_x + v u_y + \omega u_\varphi + v = 0. \tag{35}$$

The restriction (27) that follows from Coulomb's law of friction:

$$-\mu_0 \le \frac{u_x(u\,\omega^2 - u_y\omega + u_\varphi v) + (1+u_y)(v\,\omega^2 + u_x\omega - u_\varphi u)}{-u_x(v\,\omega^2 + u_x\omega - u_\varphi u) + (1+u_y)(u\,\omega^2 - u_y\omega + u_\varphi v)} \le \mu_0 \tag{36}$$

Restrictions make the problem of solving optimal control considerably more complex.

Conclusions

- The simplification of the optimal control task is also the originality of our work.
- A special contribution represents the analytical solution.
- The mass center trajectory is a deformed cycloid.
- It is shown that in the concrete case, the obtained solution represents the global minimum time.
- The manner of realizing the brachistochronic planar motion by rolling of the centroids cannot be found in other authors.
- In the case of perfectly rough surfaces, the solutions obtained based on kinematic equations represent at the same time the brachistochronic motions of.
- Also, in the case of real rough surfaces, when the maximum necessary coefficient $\boldsymbol{\mu}$ during the entire motion is smaller than the Coulomb friction coefficient $\boldsymbol{\mu_0}$.
- When this is not fulfilled, a more complex problem of optimal control has to be formulated, which will be further research subject by the authors of this paper.