Generalised holographic dark energy and novel entropies

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1 Generalized Holographic dark energy

The holographic principle originates from black hole thermodynamics and string theory and establishes a connection of the infrared cutoff of a quantum field theory, which is related to the vacuum energy, with the largest distance of this theory. Useful: cosmology! Holographic DE. Inflationary holography.

Moreover, the holographic principle is able to unify the early inflationary scenario with the late dark energy era in a covariant formalism .

HDE density is proportional to the inverse squared of the holographic cut-off (L_{IR}) which is usually assumed to be same as the particle horizon (L_{p}) or the future horizon (L_{f}). However the fundamental form of the L_{IR} is still a debatable topic in this context. Along this line, it deserves mentioning that the most generalized cut-off has been proposed in Nojiri-Odintsov,GRG38(2006)1285, where in particular, the cut-off is considered to depend upon $L_{\mathrm{IR}} = L_{\mathrm{IR}}(L_{\mathrm{p}}, \dot{L}_{\mathrm{p}}, \ddot{L}_{\mathrm{p}}, \cdots, L_{\mathrm{f}}, \dot{L}_{\mathrm{f}}, \cdots, a)$, which in turn leads to the generalized version of HDE (known as "generalized HDE"). Question:

- Does there exist suitable form(s) of L_{IR} such that various dark energy models (including the entropic DE models) can be thought to be equivalent to the generalized HDE? If so, then what will be the equivalent form(s) of L_{IR} for the respective DE models?

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In this talk based partly on Nojiri-Odintsov-Paul, Symmetry 13(2021) 6, we intend to address the above questions. Several entropic DE models, like - the Tsallis entropic DE, the Rényi entropic DE and the Sharma-Mittal entropic DE respectively. Besides the entropic DE models, the Quintessence and the Ricci-DE models will also take part in the present analysis. Interestingly, we will show that all such entropic DE, Quintessence and the Ricci-DE models are indeed equivalent with the generalized HDE, with suitable forms of the corresponding cut-offs.

Thermodynamics of Space-Time and Application to Cosmology

The entropy of the black hole is proportional to the area A of the horizon

$$S = \frac{A}{4G}\,, \quad A = 4\pi r_H^2\,, \tag{1}$$

which is called as the Bekenstein-Hawking entropy, r_H is the horizon radius. FRW equations can be also regarded as the first law of thermodynamics when we consider the Bekenstein-Hawking entropy by using the cosmological apparent horizon as a realization of the thermodynamics of space-time.

In case that, however, there are long range forces like the electro-magnetic one or gravitational one, we know that the systems are non-additive systems and the standard Boltzmann-Gibbs additive entropy should not be applied and we should generalize the entropy to the non-extensive Tsallis entropy. If we apply the Tsallis entropy to the black hole, instead of the Bekenstein-Hawking entropy, one finds ,

$$S_{\mathrm{T}} = \frac{A_0}{4G} \left(\frac{A}{A_0} \right)^{\delta}\,. \tag{2}$$

In the above expression, A_0 is a constant and δ is the new parameter that quantifies the non-extensivity. Then if we apply the Tsallis entropy by using the apparent horizon to the cosmology, the FRW equations should be modified and the modification can be regarded as the contribution from the dark energy.

In information theory, the Rényi entropy is often used as the measure of the entanglement. If we apply the Rényi entropy to the black hole, one finds

$$S_{\mathrm{R}} = \frac{A_0}{G\delta} \ln \left(1 + \frac{\delta}{4} \left(\frac{A}{A_0} \right) \right)\,. \tag{3}$$

The Rényi entropy has been also used to explain the dark energy.

Here it may be mentioned that both the Tsallis and Rényi entropy expressions belong from one-parametric entropy family; there is also a two-parametric generalized entropy which is called the Sharma-Mittal entropy ($S_{\rm SM}$) and is written as ,

$$S_{\rm SM} = \frac{A_0}{G\alpha} \left\{ \left(1 + \frac{\delta}{4} \left(\frac{A}{A_0} \right) \right)^{\frac{\alpha}{\delta}} - 1 \right\} , \tag{4}$$

where A_0 is a constant, α and δ are two independent parameters.

Dark Energy corresponding to Tsallis, Rényi, and Sharma-Mittal entropies

We assume the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with flat spacial part, whose metric is given by

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2 . \tag{5}$$

If we define the Hubble rate H by $H = \frac{\dot{a}}{a}$, the radius r_H of the cosmological horizon is given by

$$r_H = \frac{1}{H} . \tag{6}$$

Then the entropy in the region inside the cosmological horizon could be given by the Bekenstein-Hawking relation in (1). On the other hand, the flux of the energy E or the increase of the heat Q in the region is given by

$$dQ = -dE = -\frac{4\pi}{3} r_H^3 \dot{\rho} dt = -\frac{4\pi}{3H^3} \dot{\rho} dt = \frac{4\pi}{H^2} \left(\rho + p \right) dt , \tag{7}$$

where we use the conservation law: $0 = \dot{\rho} + 3H \left(\rho + p \right)$. Then by using the Hawking temperature

$$T = \frac{1}{2\pi r_H} = \frac{H}{2\pi} , \tag{8}$$

and the first law of thermodynamics $TdS = dQ$, one obtains $\dot{H} = -4\pi G \left(\rho + p \right)$ and by integrating the expression , one obtains the first FLRW equation,

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} . \tag{9}$$

Here the cosmological constant Λ appears as a constant of the integration.

Instead of the Bekenstein-Hawking entropy (1), we may use the non-extensive, the Tsallis entropy in (2). Then by applying the first law of thermodynamics to the system, instead of $\dot{H} = -4\pi G\left(\rho + p\right)$, one gets

$$\delta \left(\frac{H^2}{H_1^2} \right)^{1-\delta} \dot{H} = -4\pi G \left(\rho + p\right) \ , \tag{10}$$

on integrating which, one gets,

$$\frac{\delta}{2-\delta} H_1^2 \left(\frac{H^2}{H_1^2} \right)^{2-\delta} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \, . \tag{11}$$

Here a constant H_1 is defined by $A_0 \equiv \frac{4\pi}{H_1^2}$. Then if we define the energy density $\rho_{\rm T}$ and the pressure $p_{\rm T}$ by

$$\rho_{\rm T} = \frac{3}{8\pi G} \left(H^2 - \frac{\delta}{2-\delta} H_1^2 \left(\frac{H^2}{H_1^2} \right)^{2-\delta} \right) \ , \tag{12}$$

$$p_{\rm T} = \frac{\dot{H}}{4\pi G} \left\{ \delta \left(\frac{H^2}{H_1^2} \right)^{1-\delta} - 1 \right\} - \frac{3}{8\pi G} \left(H^2 - \frac{\delta}{2-\delta} H_1^2 \left(\frac{H^2}{H_1^2} \right)^{2-\delta} \right) \ , \tag{13}$$

respectively. It is evident that $\rho_{\rm T}$ depends on the quadratic power of the Hubble parameter and thus is symmetric with respect to the Hubble parameter. With the above forms of $\rho_{\rm T}$, $p_{\rm T}$, Eqs. (92) and (11) can be expressed as

$$\begin{aligned} \dot{H} &= -4\pi G \left[(\rho + p) + (\rho_{\rm T} + p_{\rm T}) \right] \ , \\ H^2 &= \frac{8\pi G}{3} \left(\rho_{\rm T} + \rho \right) + \frac{\Lambda}{3} \, , \end{aligned} \tag{14}$$

respectively. Therefore $\rho_{\rm T}$ and $p_{\rm T}$ represent the energy density and pressure correspond to Tsallis entropy. Consequently the respective equation of state (EoS) parameter for the Tsallis entropy is given by,

$$\omega_{\rm T} = \frac{p_{\rm T}}{\rho_{\rm T}} = -1 + 2 \left(\frac{\dot{H}}{H^2} \right) \left\{ \frac{\delta \left(\frac{H^2}{H_1^2} \right)^{1-\delta} - 1}{1 - \frac{\delta}{2-\delta} \left(\frac{H^2}{H_1^2} \right)^{1-\delta}} \right\} \tag{15}$$

It may be checked that the above expression of $\omega_{\rm T}$ leads to the conservation equation for the Tsallis entropic energy density, i.e

$$\dot{\rho}_{\rm T} + 3H\rho_{\rm T} \left(1 + \omega_{\rm T} \right) = 0 \quad . \tag{16}$$

In regard to the Rényi entropy (3), the first law of thermodynamics gives,

$$-\frac{H^3\dot{H}}{H^2+\frac{\delta}{4}H_1^2}=-\frac{4\pi G}{3}\dot{\rho}\,, \tag{17}$$

from which, we obtain

$$H^2=\frac{8\pi G}{3}\rho+\frac{\Lambda}{3}+\frac{\delta}{4}H_1^2\ln\left(\frac{H^2}{H_1^2}+\frac{\delta}{4}\right)\,. \tag{18}$$

Here the cosmological constant Λ appears as a constant of the integration again. At this stage we may define the corresponding energy density and the pressure in the following form

$$\rho_{\rm R}=\frac{3\delta}{32G}H_1^2\ln\left(\frac{H^2}{H_1^2}+\frac{\delta}{4}\right)\,, \tag{19}$$

$$p_{\rm R}=-\frac{\dot{H}}{4\pi G}\left\{\frac{1}{1+\frac{4}{\delta}\left(\frac{H^2}{H_1^2}\right)}\right\}-\frac{3\delta}{32G}H_1^2\ln\left(\frac{H^2}{H_1^2}+\frac{\delta}{4}\right)\,. \tag{20}$$

Due to the above expressions of $\rho_{\rm R}$ and $p_{\rm R}$, Eqs.(17) and (18) become similar to the usual Friedmann equations where the total energy density and total pressure are given by $\rho_{\rm eff}=\rho+\rho_{\rm R}$ and $p_{\rm eff}=p+p_{\rm R}$. Consequently, the EoS parameter corresponds to the Rényi entropy comes with the following form,

$$\omega_{\rm R}=\frac{p_{\rm R}}{\rho_{\rm R}}=-1-\frac{8}{3\pi\delta}\left(\frac{\dot{H}}{H_1^2}\right)\left\{\frac{1}{\ln\left(\frac{H^2}{H_1^2}+\frac{\delta}{4}\right)\left[1+\frac{4}{\delta}\left(\frac{H^2}{H_1^2}\right)\right]}\right\}\,. \tag{21}$$

It may be mentioned that the above expression of $\omega_{\rm R}$ obeys the conservation equation for the Rényi entropic energy density. The Rényi entropic energy density ($\rho_{\rm R}$) and the pressure ($p_{\rm R}$) can provide suitable description for the current accelerated universe and thus leads to a dark energy model.

In case of the Sharma-Mittal entropy, the first law of thermodynamics leads to the following evolution of the cosmic Hubble parameter,

$$\left(1+\frac{\delta H_1^2}{4H^2}\right)^{\frac{\alpha}{\delta}-1}\dot{H}=-4\pi G\left(\rho+p\right)\,, \tag{22}$$

integrating which, we obtain,

$$H_1^2\left(\frac{\left(\frac{\delta}{4}\right)^{\frac{\alpha}{\delta}-1}}{2-\alpha/\delta}\right)\left(\frac{H^2}{H_1^2}\right)^{2-\frac{\alpha}{\delta}}{}_2F_1\left[1-\frac{\alpha}{\delta},2-\frac{\alpha}{\delta},3-\frac{\alpha}{\delta};-\frac{4}{\delta}\left(\frac{H^2}{H_1^2}\right)\right]=\frac{8\pi G}{3}\rho+\frac{\Lambda}{3}\,, \tag{23}$$

where Λ is the constant of integration, ${}_2F_1$ is the hypergeometric function, and to get the above expression, we use the conservation equation of the matter components. Moreover, the constant H_1 is related to A_0 by $A_0 = \frac{4\pi}{H_1^2}$. Now if we define an energy density ($\rho_{\rm SM}$) and a pressure ($p_{\rm SM}$) like,

$$\rho_{\rm SM} = \frac{3}{8\pi G}\left\{H^2 - H_1^2\left(\frac{\left(\frac{\delta}{4}\right)^{\frac{\alpha}{\delta}-1}}{2-\frac{\alpha}{\delta}}\right)\left(\frac{H^2}{H_1^2}\right)^{2-\frac{\alpha}{\delta}} {}_2F_1\left[1-\frac{\alpha}{\delta}, 2-\frac{\alpha}{\delta}, 3-\frac{\alpha}{\delta}; -\frac{4}{\delta}\left(\frac{H^2}{H_1^2}\right)\right]\right\}, \tag{24}$$

$$p_{\rm SM} = \frac{\dot{H}}{4\pi G}\left\{\left(1+\frac{\delta H_1^2}{4H^2}\right)^{\frac{\alpha}{\delta}-1} - 1\right\} - \rho_{\rm SM}\,, \tag{25}$$

respectively, then Eqs. (22) and (23) can be equivalently expressed as,

$$\begin{aligned}\dot{H} &= -4\pi G\left[(\rho+p)+(\rho_{\rm SM}+p_{\rm SM})\right]\,,\\ H^2 &= \frac{8\pi G}{3}\left(\rho_{\rm SM}+\rho\right)+\frac{\Lambda}{3}\,.\end{aligned} \tag{26}$$

Thus we may argue that $\rho_{\rm SM}$ and $p_{\rm SM}$ are the energy density and the pressure coming from the cosmological description of the Sharma-Mittal entropy. Furthermore $\rho_{\rm SM}$ and $p_{\rm SM}$ are connected by the respective EoS, as given by

$$\omega_{\rm SM} = -1 + \left(\frac{\dot{H}}{3H^2}\right)\left\{\frac{\left(1+\frac{\delta H_1^2}{4H^2}\right)^{\frac{\alpha}{\delta}-1} - 1}{1-\left(\frac{\left(\frac{\delta}{4}\right)^{\frac{\alpha}{\delta}-1}}{2-\frac{\alpha}{\delta}}\right)\left(\frac{H_1^2}{H^2}\right)^{\frac{\alpha}{\delta}-1} {}_2F_1\left[1-\frac{\alpha}{\delta}, 2-\frac{\alpha}{\delta}, 3-\frac{\alpha}{\delta}; -\frac{4}{\delta}\left(\frac{H^2}{H_1^2}\right)\right]}\right\}, \tag{27}$$

where we use Eqs.(24) and (25). The above form of $\omega_{\rm SM}$ immediately confirms the conservation equation for the Sharma-Mittal entropic energy density.

Generalized Holographic Energy

In the holographic principle, the holographic energy density is proportional to the inverse squared infrared cutoff $L_{\rm IR}$, which could be related with the causality given by the cosmological horizon,

$$\rho_{\rm hol} = \frac{3c^2}{\kappa^2 L_{\rm IR}^2}\,. \tag{28}$$

Here $\kappa^2 = 8\pi G$ is the gravitational constant and c is a free parameter. The infrared cutoff $L_{\rm IR}$ is usually assumed to be the particle horizon $L_{\rm p}$ or the future event horizon $L_{\rm f}$, which are given as,

$$L_{\rm p} \equiv a\int_0^t \frac{dt}{a}\,, \quad L_{\rm f} \equiv a\int_t^\infty \frac{dt}{a}\,. \tag{29}$$

Differentiating both sides of the above expressions lead to the Hubble parameter in terms of $L_{\rm p}$, $\dot{L}_{\rm p}$ or in terms of $L_{\rm f}$, $\dot{L}_{\rm f}$ as,

$$H\left(L_{\rm p}, \dot{L}_{\rm p}\right) = \frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}, \quad H(L_{\rm f}, \dot{L}_{\rm f}) = \frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}. \tag{30}$$

a general form of the cutoff was proposed by Nojiri-Odintsov,

$$L_{\rm IR} = L_{\rm IR}\left(L_{\rm p}, \dot{L}_{\rm p}, \ddot{L}_{\rm p}, \cdots, L_{\rm f}, \dot{L}_{\rm f}, \cdots, a\right). \tag{31}$$

Actually, the other dependency of $L_{\rm IR}$, particularly on the Hubble parameter, Ricci scalar and their derivatives, can be transformed to either L_p and their derivatives or L_f and their derivatives via Eq.(86). The above cutoff could be chosen to be equivalent to a general covariant gravity model,

$$S = \int d^4\sqrt{-g}F\left(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \Box R, \Box^{-1}R, \nabla_\mu R\nabla^\mu R, \cdots\right). \tag{32}$$

The comparison of Eqs. (12) and (83) lead to the argument that the Tsallis entropic dark energy belongs to the generalized holographic dark energy family, where the corresponding infrared cutoff $L_{\rm T}$ is given by,

$$\frac{3c^2}{\kappa^2 L_{\rm T}^2} = \frac{3}{8\pi G}\left(\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2 - \frac{\delta}{2-\delta}H_1^2\left(\frac{\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2}{H_1^2}\right)^{2-\delta}\right), \tag{33}$$

in terms of $L_{\rm p}$ and its derivatives. Moreover, $L_{\rm T}$ in terms of the future horizon and its derivatives comes by the following way,

$$\frac{3c^2}{\kappa^2 L_{\rm T}^2} = \frac{3}{8\pi G}\left(\left(\frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}\right)^2 - \frac{\delta}{2-\delta}H_1^2\left(\frac{\left(\frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}\right)^2}{H_1^2}\right)^{2-\delta}\right). \tag{34}$$

Here we would like to determine the EoS parameter of the holographic energy density corresponds to the cut-off $L_{\rm T}$, in particular of $\rho_{\rm hol} = 3c^2/\left(\kappa^2 L_{\rm T}^2\right)$. In this regard, the conservation equation of $\rho_{\rm hol}$ immediately yields the respective EoS parameter (symbolized by $\Omega_{\rm hol}^{(T)}$) as,

$$\Omega_{\rm hol}^{(T)} = -1 - \left(\frac{2}{3HL_{\rm T}}\right)\frac{dL_{\rm T}}{dt}, \tag{35}$$

where $L_{\rm T}$ is obtained in Eq.(33) (or Eq.(34)) and the superscript 'T' in the above expression denotes the EoS parameter corresponds to the holographic cut-off $L_{\rm T}$. Due to Eq.(86), the above form of $\Omega^{(T)}_{\rm hol}$ is equivalent to the EoS of the Tsallis entropic energy density presented in Eq.(15), i.e $\Omega^{(T)}_{\rm hol} \equiv \omega_{\rm T}$. Such equivalence, along with the fact that the Tsallis entropic energy density provides a viable dark energy model, lead to the argument that the holographic energy density coming from the cut-off $L_{\rm T}$ is also able to produce a viable dark energy epoch at our current universe.

Similarly, by comparing (95) and (83), the infrared cutoff $L_{\rm R}$ corresponding to the Rényi entropy is given by

$$\frac{3c^2}{\kappa^2 L_{\rm R}^2} = \frac{3\delta}{32G} H_1^2 \ln\left(\frac{1}{H_1^2}\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2 + \frac{\delta}{4}\right) = \frac{3\delta}{32G} H_1^2 \ln\left(\frac{1}{H_1^2}\left(\frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}\right)^2 + \frac{\delta}{4}\right), \tag{36}$$

where, once again, we use Eq. (86). The first expression of Eq. (36) gives the $L_{\rm R}$ in terms of $L_{\rm p}$ and its derivatives, while the second one represents the same in terms of $L_{\rm f}$ and its derivatives. Once again, the conservation equation of the holographic energy density $\rho_{\rm hol} = 3c^2/\left(\kappa^2 L_{\rm R}^2\right)$ leads to the corresponding EoS parameter ($\Omega^{(R)}_{\rm hol}$) as,

$$\Omega^{(R)}_{\rm hol} = -1 - \left(\frac{2}{3HL_{\rm R}}\right)\frac{dL_{\rm R}}{dt}\ , \tag{37}$$

where $L_{\rm R}$ is given in Eq.(36). Thereby, since the Rényi entropic energy density suitably describes the current acceleration of our universe, we may argue that the holographic energy density coming from $L_{\rm R}$ is able to produce the late time cosmic acceleration.

Finally Eqs. (24) and (83) clearly argue that the Sharma-Mittal entropic dark energy can also be thought as one of the candidates of the generalized dark energy family, where the corresponding cut-off ($L_{\rm SM}$) is given by,

$$\begin{aligned}\frac{3c^2}{\kappa^2 L_{\rm SM}^2} =& \frac{3}{8\pi G}\left\{\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2 - H_1^2\left(\frac{\left(\frac{\delta}{4}\right)^{\frac{\alpha}{\delta}-1}}{2-\alpha/\delta}\right)\left(\frac{\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2}{H_1^2}\right)^{2-\frac{\alpha}{\delta}}\right.\\ &\left.\times {}_2F_1\left[1-\frac{\alpha}{\delta}, 2-\frac{\alpha}{\delta}, 3-\frac{\alpha}{\delta}; -\frac{4}{\delta}\left(\frac{\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2}{H_1^2}\right)\right]\right\},\end{aligned} \tag{38}$$

in terms of the particle horizon and its derivatives. Similarly, the $L_{\rm SM}$ in terms of the future horizon and its derivatives is given by,

$$\frac{3c^2}{\kappa^2 L_{\mathrm{SM}}^2} = \frac{3}{8\pi G}\left\{\left(\frac{\dot{L}_{\mathrm{f}}}{L_{\mathrm{f}}} + \frac{1}{L_{\mathrm{f}}}\right)^2 - H_1^2\left(\frac{\left(\frac{\delta}{4}\right)^{\frac{\alpha}{\delta}-1}}{2-\frac{\alpha}{\delta}}\right)\left(\frac{\left(\frac{\dot{L}_{\mathrm{f}}}{L_{\mathrm{f}}} + \frac{1}{L_{\mathrm{f}}}\right)^2}{H_1^2}\right)^{2-\frac{\alpha}{\delta}}\right.$$
$$\left.\times {}_2F_1\left[1-\frac{\alpha}{\delta}, 2-\frac{\alpha}{\delta}, 3-\frac{\alpha}{\delta}; -\frac{4}{\delta}\left(\frac{\left(\frac{\dot{L}_{\mathrm{f}}}{L_{\mathrm{f}}} + \frac{1}{L_{\mathrm{f}}}\right)^2}{H_1^2}\right)\right]\right\} . \tag{39}$$

Furthermore using the conservation relation of $\rho_{\mathrm{hol}} = 3c^2/\left(\kappa^2 L_{\mathrm{SM}}^2\right)$, we determine the EoS parameter ($\Omega_{\mathrm{hol}}^{(SM)}$) corresponds to the holographic energy density coming from the cut-off L_{SM} as,

$$\Omega_{\mathrm{hol}}^{(SM)} = -1 - \left(\frac{2}{3HL_{\mathrm{SM}}}\right)\frac{dL_{\mathrm{SM}}}{dt} , \tag{40}$$

where L_{SM} is given in Eq.(38) (or in Eq.(39)). In effect of Eq.(86), it is evident that the above form of $\Omega_{\mathrm{hol}}^{(SM)}$ is equivalent to the EoS of the Sharma-Mittal entropic energy density of Eq.(27), i.e $\Omega_{\mathrm{hol}}^{(SM)} = \omega_{\mathrm{SM}}$. Due to this equivalence, we may argue that the holographic energy density $\rho_{\mathrm{hol}} = 3c^2/\left(\kappa^2 L_{\mathrm{SM}}^2\right)$ can produce the late time acceleration of our universe.

Therefore the dark energy models coming from the Tsallis entropy, the Rényi entropy and the Sharma-Mittal entropy can be thought as different candidates of the generalized holographic dark energy family, where the respective infrared cutoffs are given by Eq. (33) to Eq. (39) respectively.

Quintessence dark energy

Scalar field dark energy models! So far, a wide amount of scalar field dark energy models have been proposed, these include Quintessence, Phantoms, K-essence, Tachyon, Dilatonic dark energy etc.

In this section, we consider the Quintessence dark energy (QDE) model and show that QDE is equivalent to the generalized holographic dark energy model where $L_{\mathrm{IR}} = L_{\mathrm{IR}}\left(L_{\mathrm{p}}, \dot{L}_{\mathrm{p}}, \ddot{L}_{\mathrm{p}}, L_{\mathrm{f}}, \dot{L}_{\mathrm{f}}, \ddot{L}_{\mathrm{f}}\right)$. The QDE action is given by,

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] , \tag{41}$$

where ϕ is the Quintessence scalar field and $V(\phi)$ is its potential. The Quintessence potential has the following form,

$$V(\phi) = V_0 \exp\left[-\sqrt{\frac{16\pi G}{p}}\phi\right], \tag{42}$$

with V_0 and p are constants. The Quintessence model with the above exponential potential has been extensively studied and it was shown that the potential of Eq.(42) leads to a viable dark energy model in respect to SNIa, BAO and H(z) observations. However the most stringent constraints on the dark energy EoS parameter (ω_{Q}) comes from the BAO observations, in particular $-1 < \omega_{\mathrm{Q}} < -0.85$.

The FLRW equations correspond to the action (41) are,

$$\begin{aligned} H^2 =& \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \\ \dot{H} =& -4\pi G\dot{\phi}^2, \end{aligned} \tag{43}$$

The first FLRW equation immediately leads to the Quintessence energy density as,

$$\rho_{\mathrm{Q}} = \frac{1}{2}\dot{\phi}^2 + V(\phi) = -\frac{\dot{H}}{8\pi G} + V(\phi), \tag{44}$$

where in the second line, we use $\dot{H} = -4\pi G\dot{\phi}^2$. The exponential form of the Quintessence potential (see Eq. (42)) allows the following solutions of the Hubble parameter and the scalar field as,

$$H = \frac{p}{t} \quad \text{and} \quad \phi(t) = 2\sqrt{\frac{p}{16\pi G}}\ln\left(\frac{t}{t_0}\right), \tag{45}$$

respectively. Here V_0, p, and t_0 are related by the following constraint equation,

$$3p - 1 = V_0 t_0^2\left(\frac{8\pi G}{p}\right). \tag{46}$$

Furthermore the evolution of the Hubble parameter clearly indicates that in order to get an accelerating expansion of the universe, the parameter p is constrained to be $p > 1$. By using Eqs. (42) and (45), we can express the Quintessence potential in terms of the Hubble parameter as follows,

$$V(\phi) = \left(3 - \frac{1}{p}\right)\frac{H^2}{8\pi G}. \tag{47}$$

Plugging back the above expression into Eq. (44), we get $\rho_{\rm Q}$ in terms of H and $\dot{H}$ as,

$$\rho_{\rm Q} = \frac{1}{8\pi G}\left\{\left(3-\frac{1}{p}\right)H^2 - \dot{H}\right\} . \tag{48}$$

Furthermore the pressure in the present context is given by,

$$p_{\rm Q} = -\frac{\dot{H}}{8\pi G} - V(\phi) = -\frac{1}{8\pi G}\left\{\left(3-\frac{1}{p}\right)H^2 + \dot{H}\right\} , \tag{49}$$

which, along with Eq.(48) immediately leads to the corresponding EoS parameter as,

$$\omega_{\rm Q} = -\frac{\left\{\left(3-\frac{1}{p}\right)H^2 + \dot{H}\right\}}{\left\{\left(3-\frac{1}{p}\right)H^2 - \dot{H}\right\}} . \tag{50}$$

Having set the stage, now we are in a position to show the equivalence between QDE and generalized holographic dark energy model. The comparison of Eqs. (48) and (83) immediately lead to the equivalent holographic cut-off $(L_{\rm Q})$ corresponds to the QDE as follows,

$$\begin{aligned}\frac{3c^2}{\kappa^2 L_{\rm Q}^2} =& \frac{1}{8\pi G}\left\{\left(3-\frac{1}{p}\right)\left(\frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\right)^2 - \left(\frac{\ddot{L}_{\rm p}}{L_{\rm p}} - \frac{\dot{L}_{\rm p}^2}{L_{\rm p}^2} + \frac{\dot{L}_{\rm p}}{L_{\rm p}^2}\right)\right\}\\ =& \frac{1}{8\pi G}\left\{\left(3-\frac{1}{p}\right)\left(\frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}\right)^2 - \left(\frac{\ddot{L}_{\rm f}}{L_{\rm f}} - \frac{\dot{L}_{\rm f}^2}{L_{\rm f}^2} - \frac{\dot{L}_{\rm f}}{L_{\rm f}^2}\right)\right\} .\end{aligned} \tag{51}$$

Thereby the QDE can be equivalently mapped to the generalized holographic dark energy model where the cut-off is the function of $L_{\rm p}$, $\dot{L}_{\rm p}$, $\ddot{L}_{\rm p}$ or the function of $L_{\rm f}$, $\dot{L}_{\rm f}$, $\ddot{L}_{\rm f}$. Furthermore, the EoS parameter $(\Omega_{\rm hol}^{(Q)})$ corresponds to the hologrphic cut-off $L_{\rm Q}$ is given by,

$$\Omega_{\rm hol}^{(Q)} = -1 - \left(\frac{2}{3HL_{\rm Q}}\right)\frac{dL_{\rm Q}}{dt} , \tag{52}$$

where $L_{\rm Q}$ is shown above. Clearly, in accordance of Eq.(86), $\Omega_{\rm hol}^{(Q)}$ becomes equivalent to the $\omega_{\rm Q}$ of Eq.(50). Such equivalence leads to the fact that similar to the Quintessence energy density, the holographic energy density coming from the cut-off $L_{\rm Q}$ also provides a good dark energy model of our universe.

The same approach to show that Ricci DE is just one representative of generalised HDE! Using same technique with holographic inflation we can propose unified holographic inflation-dark energy evolution. It was developed in detail for number of scenarios in [Nojiri-Odintsov-Oikonomou-Paul, PRD102(2020)023540].

Conclusion: any Dark Energy model maybe shown to be representative of generalised holographic DE. Any specific inflationary model maybe mapped with generalised holographic inflation too.

2 How fundamental is entropy: on the way to unique universal entropy construct

Nojiri-Odintsov-Faraoni, PRD 105(2022)4.

Classical thermodynamics: entropy is as a unique, universal, and fundamental quantity playing one of the most important roles in physics. Not so! Not fundamental, not unique. It depends on physical system under consideration.

Variety of entropies exists in many classical and quantum systems. Still more maybe expected!!!

In the 1970s: black holes are not cold objects but have entropy and temperature. Bekenstein's association of entropy with black holes, proportional to the black hole horizon area , remained odd and inconclusive until Hawking discovered that the Schwarzschild black hole (and, by extension, all black holes) radiate quanta of quantum fields living on that spacetime, emitting a blackbody spectrum at a temperature $T_{\mathrm{H}} = \frac{1}{8\pi GM}$, where M is the black hole mass. The discovery of the Hawking temperature made sense of Bekenstein's black hole entropy and paved the way for the development of black hole thermodynamics.

One puzzling feature of the Bekenstein-Hawking entropy was, from its beginnings, that it is not proportional to the black hole volume, as familiar in classical thermodynamics, but rather it is proportional to the black hole horizon area. In classical thermodynamics, the entropy of a system is proportional to its mass and its volume and is an extensive and additive quantity; the fundamental reason why black hole entropy is instead non-extensive remains shrouded in mystery. Given the elusive nature of the origin of this entropy, it is not surprising that recent literature contemplates alternatives, replacing the Bekenstein-Hawking entropy with other constructs based on non-extensive statistics, including the Rényi and Tsallis non-extensive entropies (a better terminology would "non-additive" entropies). Since entropy, temperature, internal energy, and heat transferred are related by the first law of thermodynamics, changing the notion of entropy entails changes in these other quantities, usually jeopardizing the first law.

Horizons are not a prerogative of black holes but appear also in cosmology, hence horizon thermodynamics was extended to cosmological horizons. Many scenarios of dark energy and modified gravity have been proposed and are being tested and/or constrained as newer cosmological observations become available. Among the many scenarios advanced in the cosmology literature, the holographic dark energy proposal is directly related to entropy. Therefore, replacing the notion of entropy used in physics has a direct impact on this scenario.

Possible generalizations of known entropies

The Bekenstein-Hawking entropy is

$$\mathcal{S} = \frac{A}{4G}\,, \tag{53}$$

where $A \equiv 4\pi {r_\mathrm{h}}^2$ is the area of the horizon and r_h is the horizon radius (using the areal radius as the radial coordinate). This proposal, however, is not unique. Indeed, depending on the system under consideration, different entropies may be introduced. Let us recall some of the entropy concepts proposed thus far.

- The Tsallis entropy appears in the study of non-extensive statistics for systems with long range interactions, in which the partition function diverges and the standard Boltzmann-Gibbs entropy becomes inadequate; it is

  $$\mathcal{S}_\mathrm{T} = \frac{A_0}{4G}\left(\frac{A}{A_0}\right)^{\delta}\,, \tag{54}$$

  where A_0 is a constant with the dimensions of an area and δ is a dimensionless parameter that quantifies the non-extensivity. The standard Bekenstein-Hawking entropy (53) is recovered for $\delta = 1$.

- The Rényi entropy is defined as

  $$\mathcal{S}_\mathrm{R} = \frac{1}{\alpha}\ln\left(1 + \alpha\mathcal{S}\right) \tag{55}$$

where $\mathcal{S}$ is identified with the Bekenstein-Hawking entropy (53), and contains a parameter α. The Rényi entropy was proposed as an index specifying the amount of information and, originally, had no relation with the statistics of physical systems.

- The Sharma-Mittal entropy is

$$\mathcal{S}_{\mathrm{SM}} = \frac{1}{R}\left[\left(1+\delta\,\mathcal{S}_{\mathrm{T}}\right)^{R/\delta} - 1\right] \tag{56}$$

where $\mathcal{S}_{\mathrm{T}}$ is the Tsallis entropy, while R and δ are free phenomenological parameters to be determined by the best-fitting of experimental data. The Sharma-Mittal entropy can be seen as a combination of the Rényi and Tsallis entropies.

- The Barrow entropy is

$$\mathcal{S}_{\mathrm{B}} = \left(\frac{A}{A_{\mathrm{Pl}}}\right)^{1+\Delta/2} ; \tag{57}$$

here A is the usual black hole horizon area and $A_{\mathrm{Pl}} \equiv 4G$ is the Planck area. Formally, the Barrow entropy resembles the Tsallis non-extensive entropy but the physical principles underlying its introduction are radically different. The Barrow entropy was proposed as a toy model for the possible effects of quantum gravitational spacetime foam. The quantum-gravitational deformation is quantified by the new exponent Δ. The Barrow entropy reduces to the standard Bekenstein-Hawking entropy in the limit $\Delta \to 0$, while $\Delta = 1$ corresponds to maximal deformation.

- The Kaniadakis entropy

$$\mathcal{S}_{\mathrm{K}} = \frac{1}{K}\sinh\left(K\mathcal{S}\right) , \tag{58}$$

reproduces the Bekenstein-Hawking entropy in the limit $K \to 0$ of its parameter K. It can be regarded as a generalization of the Boltzmann-Gibbs entropy arising in relativistic statistical systems.

- Non-extensive statistical mechanics in Loop Quantum Gravity gives the entropy

$$\mathcal{S}_q = \frac{1}{1-q}\left[\mathrm{e}^{(1-q)\Lambda(\gamma_0)\mathcal{S}} - 1\right] , \tag{59}$$

where the entropic index q quantifies how the probability of frequent events is enhanced relatively to infrequent ones,

$$\Lambda(\gamma_0) = \frac{\ln 2}{\sqrt{3}\,\pi\gamma_0} \,, \tag{60}$$

and γ_0 is the Barbero-Immirzi parameter, which is usually assumed to take one of the two values $\frac{\ln 2}{\pi\sqrt{3}}$ or $\frac{\ln 3}{2\pi\sqrt{2}}$, depending on the gauge group used in Loop Quantum Gravity. However, γ_0 is a free parameter in scale-invariant gravity. With the first choice of γ_0, $\Lambda(\gamma_0)$ becomes unity and the entropy (59) reduces to the Bekenstein-Hawking one in the limit $q \to 1$, which corresponds to extensive statistical mechanics.

The above entropies share the following properties:

1. *Generalized third law:* All these entropies vanish when the Bekenstein-Hawking entropy vanishes. In the third law of standard thermodynamics for closed systems in thermodynamic equilibrium, the quantity $\mathrm{e}^{\mathcal{S}}$ expresses the number of states, or the volume of these states, and therefore the entropy $\mathcal{S}$ vanishes when the temperature does because the ground (vacuum) state should be unique. By contrast, the Bekenstein-Hawking entropy $\mathcal{S}$ diverges when the temperature T vanishes and it goes to zero at infinite temperature. However, requiring any generalized entropy to vanish when the Bekenstein-Hawking entropy $\mathcal{S}$ vanishes could be a natural requirement.

2. *Monotonically increasing functions:* All the above entropies are monotonically increasing functions of the Bekenstein-Hawking entropy $\mathcal{S}$.

3. *Positivity:* All the above entropies are positive, as is the Bekenstein-Hawking entropy (53). This is natural because $\mathrm{e}^{\mathcal{S}}$ corresponds to the number of states (or to the volume of these states), which is greater than unity.

4. *Bekenstein-Hawking limit:* All the above entropies reduce to the Bekenstein-Hawking entropy (53) in an appropriate limit.

In the preceding expressions, all entropies are functions of the Bekenstein-Hawking entropy (53). In this sense, the most general entropy $\mathcal{S}_\mathrm{G}$ would be a function of the Bekenstein-Hawking entropy $\mathcal{S}$,

$$\mathcal{S}_\mathrm{G} = \mathcal{S}_\mathrm{G}\left(\mathcal{S}\right) \,, \tag{61}$$

subject to certain natural requirements: we require the general entropy $\mathcal{S}_{\rm G}$ to possess the above properties.

An example of such an entropy construct containing six parameters $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ could be

$$\mathcal{S}_{\rm G}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right) = \frac{1}{\alpha_+ + \alpha_-}\left[\left(1 + \frac{\alpha_+}{\beta_+}\,\mathcal{S}^{\gamma_+}\right)^{\beta_+} - \left(1 + \frac{\alpha_-}{\beta_-}\,\mathcal{S}^{\gamma_-}\right)^{-\beta_-}\right]\,, \tag{62}$$

where we assume all the parameters $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ to be positive. First, we show that the entropy $\mathcal{S}_{\rm G}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ reduces to the entropies (54), (55), (56), (57), (58), and (59) already presented for appropriate choices of the parameter values.

- In the limit $\alpha_+ = \alpha_- \to 0$, the choice $\gamma_+ = \gamma_- \equiv \gamma$ gives

  $$\mathcal{S}_{\rm G}\left(\alpha_\pm \to 0, \beta_\pm, \gamma\right) \to \mathcal{S}^\gamma\,. \tag{63}$$

  If we further choose $\gamma = \delta$ or $\gamma = 1 + \Delta/2$, the Tsallis entropy (54) or the Barrow entropy (57) are reproduced, respectively.

- The parameter choice $\alpha_- = 0$ yields

  $$\mathcal{S}_{\rm G}\left(\alpha_+, \alpha_- = 0, \beta_\pm, \gamma_+ = 1, \gamma_-\right) = \frac{1}{\alpha_+}\left[\left(1 + \frac{\alpha_+}{\beta_+}\mathcal{S}^{\gamma_+}\right)^{\beta_+} - 1\right]\,. \tag{64}$$

  Then, writing $\alpha_+ = R$, $\beta_+ = R/\delta$, and $\gamma_+ = \delta$, one obtains the Sharma-Mittal entropy (56).

- In Eq. (64), if we further take the limit $\alpha_+ \to 0$ simultaneously with $\beta_+ \to 0$ keeping $\alpha \equiv \alpha_+/\beta_+$ finite, and we choose $\gamma_+ = 1$, we obtain

  $$\begin{aligned}
  &\mathcal{S}_{\rm G}\left(\alpha_+ \to 0, \alpha_- = 0, \beta_+ \to 0, \beta_-, \gamma_+ = 1, \gamma_-; \alpha \equiv \frac{\alpha_+}{\beta_+}\ \text{finite}\right) \\
  &\to \frac{1}{\alpha_+}\left[{\rm e}^{\beta_+ \ln\left(1 + \frac{\alpha_+}{\beta_+}\mathcal{S}\right)} - 1\right] \simeq \frac{1}{\alpha_+}\left[1 + \beta_+ \ln\left(1 + \frac{\alpha_+}{\beta_+}\mathcal{S}\right) - 1\right] \\
  &= \frac{\beta_+}{\alpha_+}\ln\left(1 + \frac{\alpha_+}{\beta_+}\mathcal{S}\right) \equiv \frac{1}{\alpha}\ln\left(1 + \alpha\mathcal{S}\right)\,,
  \end{aligned} \tag{65}$$

  which reproduces the Rényi entropy (55).

- Taking the limit $\beta_\pm \to 0$, choosing $\gamma_\pm = 1$, and writing $\alpha_\pm = K$, the general entropy (62) reduces to the Kaniadakis one (58),

$$\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm = K, \beta_\pm \to 0, \gamma_\pm = 1\right) \to \frac{1}{2K}\left(\mathrm{e}^{K\mathcal{S}} - \mathrm{e}^{-K\mathcal{S}}\right) = \frac{1}{K}\sinh\left(K\mathcal{S}\right) . \tag{66}$$

- Finally, taking $\alpha_- = 0$ and $\gamma_+ = 1$ in the generalized entropy (62), one obtains

$$\mathcal{S}_G = \frac{1}{\alpha_+}\left[\mathrm{e}^{\beta_+ \ln\left(1+\frac{\alpha_+}{\beta_+}\mathcal{S}\right)} - 1\right] \tag{67}$$

and the further limit $\beta_+ \to +\infty$ in conjunction with $\alpha = 1 - q$ yields

$$\mathcal{S}_G \approx \frac{1}{1-q}\left[\mathrm{e}^{(1-q)\mathcal{S}} - 1\right] \tag{68}$$

corresponding to $\Lambda(\gamma_0) = 1$ in the Loop Quantum Gravity entropy (59), and which reduces to the Bekenstein-Hawking entropy $\mathcal{S}$ as $q \to 1$.

It is straightforward to check that the entropy $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ in Eq. (62) satisfies the generalized third law, that is, $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right) \to 0$ when $\mathcal{S} \to 0$. The entropy $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ is a monotonically increasing function of $\mathcal{S}$ because both $\left(1 + \frac{\alpha_+}{\beta_+}\mathcal{S}^{\gamma_+}\right)^{\beta_+}$ and $-\left(1 + \frac{\alpha_-}{\beta_-}\mathcal{S}^{\gamma_-}\right)^{-\beta_-}$ are monotonically increasing functions of $\mathcal{S}$, given that all the parameters $\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ are assumed to be positive, and their sum is also monotonically increasing. Positivity is satisfied because $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right) = 0$ when $\mathcal{S} = 0$ and $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ is a strictly increasing function of $\mathcal{S}$.

It is clear that there exists a limit of $\mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right)$ to the Bekenstein-Hawking entropy because $\mathcal{S}_{\mathrm{G}}$ reduces to the entropies (54), (55), (56), (57), (58), and (59), which have the required limiting behaviour. More explicitly, we have

$$\lim_{\alpha_\pm \to 0} \mathcal{S}_{\mathrm{G}}\left(\alpha_\pm, \beta_\pm, \gamma_\pm\right) = \mathcal{S} . \tag{69}$$

We may also consider the three-parameter entropy-like quantity

$$\mathcal{S}_{\mathrm{G}}\left(\alpha, \beta, \gamma\right) = \frac{1}{\gamma}\left[\left(1 + \frac{\alpha}{\beta}\mathcal{S}\right)^{\beta} - 1\right] , \tag{70}$$

where we assume again the parameters (α, β, γ) to be positive. When γ and α coincide, the expression (70) reduces to the Sharma-Mittal entropy (56) with $\mathcal{S}_{\mathrm{T}} = \mathcal{S}$, that is, $\delta = 1$. By writing $\gamma = (\alpha/\beta)^{\beta}$, the limit $\alpha \to \infty$ yields

$$\lim_{\alpha \to \infty} \mathcal{S}_{\mathrm{G}}\left(\alpha, \beta, \gamma = \left(\frac{\alpha}{\beta}\right)^{\beta}\right) = \mathcal{S}^{\beta} . \tag{71}$$

The choices $\beta = \delta$ and $\beta = 1 + \Delta/2$ give the Tsallis entropy (54) and the Barrow entropy (57), respectively. If, instead, we consider the limit in which $\alpha \to 0$ and $\beta \to 0$ simultaneously while keeping α/β finite, as in Eq. (65), we obtain the Rényi entropy (55) by replacing α/β with α and choosing $\gamma = \alpha$,

$$\mathcal{S}_{\mathrm{G}}\left(\alpha \to 0, \beta \to 0, \gamma; \frac{\alpha}{\beta} \text{ finite}\right) \to \frac{1}{\gamma} \ln\left(1 + \frac{\alpha}{\beta}\, \mathcal{S}\right) = \frac{1}{\alpha} \ln\left(1 + \alpha \mathcal{S}\right) . \quad (72)$$

Another possibility consists of taking the limit $\beta \to \infty$ in conjunction with $\gamma = \alpha$, which leads to the new type of expression

$$\mathcal{S}_{\mathrm{G}}\left(\alpha, \beta \to \infty, \gamma\right) \to \frac{1}{\gamma}\left(\mathrm{e}^{\alpha \mathcal{S}} - 1\right) . \quad (73)$$

It is again straightforward to check that (70) satisfies all the conditions characterizing the generalized third law: monotonically increasing function of $\mathcal{S}$, positivity, and Bekenstein-Hawking limit.

To recap, we have proposed two new examples of entropy that may be valid for the description of certain physical systems, which we have not yet discussed. Eventually, several additional proposals for even more general entropies can be conceived. However, we still lack a physical principle selecting an entropy as unique and universal, perhaps containing many parameters depending on various quantities.

Black hole thermodynamics with 3-parameter generalized entropy

It is interesting to see what happens when the generalized entropy (61) is ascribed to the prototypical black hole, given by the Schwarzschild geometry

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{(2)}^2 , \qquad f(r) = 1 - \frac{2GM}{r} , \quad (74)$$

where M is the black hole mass and $d\Omega_{(2)}^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the line element on the unit two-sphere. The black hole event horizon is located at the Schwarzschild radius

$$r_{\mathrm{H}} = 2GM . \quad (75)$$

Studying quantum field theory on the spacetime with this horizon, Hawking discovered that the Schwarzschild black hole radiates with a blackbody spectrum at the temperature

$$T_{\mathrm{H}} = \frac{1}{8\pi GM} . \quad (76)$$

As explained in general below, if we assume that the mass M coincides with the thermodynamical energy, then the temperature obtained from the thermodynamical law is different from the Hawking temperature, a contradiction for observers detecting Hawking radiation. Alternatively, if the Hawking temperature $T_{\rm H}$ is identified with the physical black hole temperature, the obtained thermodynamical energy differs from the Schwarzschild mass M even for the Tsallis entropy or the Rényi entropy, which seems to imply a breakdown of energy conservation.

If the mass M coincides with the thermodynamical energy E of the system due to energy conservation, as in, in order for this system to be consistent with the thermodynamical equation $d\mathcal{S}_G = dE/T$ one needs to define the generalized temperature $T_{\rm G}$ as

$$\frac{1}{T_{\rm G}} \equiv \frac{d\mathcal{S}_{\rm G}}{dM} \tag{77}$$

which is, in general, different from the Hawking temperature $T_{\rm H}$. For example, in the case of the entropy (70), one has

$$\frac{1}{T_{\rm G}} = \frac{\alpha}{\gamma}\left(1+\frac{\alpha}{\beta}\,\mathcal{S}\right)^{\beta-1}\frac{d\mathcal{S}}{dM} = \frac{\alpha}{\gamma}\left(1+\frac{\alpha}{\beta}\,\mathcal{S}\right)^{\beta-1}\frac{1}{T_{\rm H}}\,, \tag{78}$$

where

$$\mathcal{S} = \frac{A}{4G} = 4\pi G M^2 = \frac{1}{16\pi G {T_{\rm H}}^2}\,. \tag{79}$$

Because $\frac{\alpha}{\gamma}\left(1+\frac{\alpha}{\beta}\,\mathcal{S}\right)^{\beta-1} \neq 1$, it is necessarily $T_{\rm G}\neq T_{\rm H}$. Since the Hawking temperature (76) is the temperature perceived by observers detecting Hawking radiation, the generalized temperature $T_{\rm G}$ in (78) cannot be a physically meaningful temperature.

In Eq. (77), assuming that the thermodynamical energy E is the black hole mass M leads to an unphysical result. As an alternative, assume that the thermodynamical temperature T coincides with the Hawking temperature $T_{\rm H}$ instead of assuming $E=M$. This assumption leads to

$$dE_{\rm G} = T_{\rm H}\, d\mathcal{S}_{\rm G} = \frac{d\mathcal{S}_{\rm G}}{d\mathcal{S}}\,\frac{d\mathcal{S}}{\sqrt{16\pi G\mathcal{S}}} \tag{80}$$

which, in the case of Eq. (70), yields

$$dE_{\rm G} \;=\; \frac{\alpha}{\gamma}\left(1+\frac{\alpha}{\beta}\,\mathcal{S}\right)^{\beta-1}\frac{d\mathcal{S}}{\sqrt{16\pi G\mathcal{S}}}$$

$$= \frac{\alpha}{\gamma\sqrt{16\pi G}}\left[\mathcal{S}^{-1/2} + \frac{\alpha(\beta-1)}{\beta}\mathcal{S}^{1/2} + \mathcal{O}\left(\mathcal{S}^{3/2}\right)\right] . \tag{81}$$

The integration of Eq. (81) gives

$$\begin{aligned} E_{\mathrm{G}} &= \frac{\alpha}{\gamma\sqrt{16\pi G}}\left[2\mathcal{S}^{1/2} + \frac{2\alpha(\beta-1)}{3\beta}\mathcal{S}^{3/2} + \mathcal{O}\left(\mathcal{S}^{5/2}\right)\right] \\ &= \frac{\alpha}{\gamma}\left[M + \frac{4\pi G\alpha(\beta-1)}{3\beta}M^3 + \mathcal{O}\left(M^5\right)\right] , \end{aligned} \tag{82}$$

where the integration constant is determined by the condition that $E_{\mathrm{G}} = 0$ when $M = 0$. Even when $\alpha = \gamma$, due to the correction $\frac{4\pi G\alpha(\beta-1)}{3\beta}M^3$, the expression (82) of the thermodynamical energy E_{R} obtained differs from the black hole mass M, $E_{\mathrm{G}} \neq E$, which seems unphysical.

Holographic cosmology with generalized entropy

The density of the holographic dark energy (HDE) is proportional to the square of the inverse holographic cutoff L_{IR},

$$\rho_{\mathrm{hol}} = \frac{3C^2}{\kappa^2 {L_{\mathrm{IR}}}^2} , \tag{83}$$

where C is a free parameter. The holographic cutoff L_{IR} is usually assumed to be the same as the particle horizon L_{p} or the future horizon L_{f}. No compelling argument has been proposed thus far for choosing this quantity, hence the most general cutoff was proposed by Nojiri-Odintsov,2006. In this proposal, the cutoff is assumed to depend upon $L_{\mathrm{IR}} = L_{\mathrm{IR}}(L_{\mathrm{p}}, \dot{L}_{\mathrm{p}}, \ddot{L}_{\mathrm{p}}, \cdots, L_{\mathrm{f}}, \dot{L}_{\mathrm{f}}, \cdots, a)$, which in turn leads to the generalized version of HDE known as "generalized HDE" . In the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe described by the line element

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3}\left(dx^i\right)^2 \tag{84}$$

with scale factor $a(t)$ in comoving coordinates (t, x, y, z), one might speculate that the generalized HDE originates from one of several kinds of entropies associated with the cosmological horizon. In the FLRW spacetime (84), the particle horizon L_{p} and the future event horizon L_{f} are defined as

$$L_{\mathrm{p}} \equiv a(t)\int_0^t \frac{dt'}{a(t')} , \qquad L_{\mathrm{f}} \equiv a(t)\int_t^\infty \frac{dt'}{a(t')} , \tag{85}$$

respectively, when these integrals converge. Differentiating both sides of these definitions leads to the expressions of the Hubble function in terms of $L_{\rm p}$, $\dot{L}_{\rm p}$ or of $L_{\rm f}$, $\dot{L}_{\rm f}$ (where an overdot denotes differentiation with respect to the comoving time t)

$$H\left(L_{\rm p}, \dot{L}_{\rm p}\right) = \frac{\dot{L}_{\rm p}}{L_{\rm p}} - \frac{1}{L_{\rm p}}\,, \qquad H(L_{\rm f}, \dot{L}_{\rm f}) = \frac{\dot{L}_{\rm f}}{L_{\rm f}} + \frac{1}{L_{\rm f}}\,, \tag{86}$$

where the Hubble rate is $H \equiv \dot{a}/a$.

As argued, *e.g.*, the standard Einstein-Friedmann equations can be derived from the Bekenstein-Hawking entropy (53). The physical radius of the cosmological horizon in spatially flat FLRW universes is

$$r_{\rm H} = \frac{1}{H}\,, \tag{87}$$

which tells us that the entropy inside this horizon can be given by the Bekenstein-Hawking entropy (53) with the identification $A \equiv 4\pi {r_{\rm h}}^2 = 4\pi {r_{\rm H}}^2$. Because the incremental change of the energy E, or the increase of the heat Q, contained in this region is given by

$$dQ = -dE = -\frac{4\pi}{3}\, r_{\rm H}^3\, \dot{\rho}\, dt = -\frac{4\pi}{3H^3}\, \dot{\rho}\, dt = \frac{4\pi}{H^2}\left(\rho + P\right) dt \tag{88}$$

(where we used the conservation law $\dot{\rho} + 3H\left(\rho + P\right) = 0$), by using the Gibbons-Hawking temperature

$$T = \frac{1}{2\pi r_{\rm H}} = \frac{H}{2\pi} \tag{89}$$

and the first law of thermodynamics $TdS = dQ$, we obtain

$$\dot{H} = -4\pi G\left(\rho + P\right)\,. \tag{90}$$

The integration of Eq. (90) leads to the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}\,, \tag{91}$$

where the integration constant corresponds to the cosmological constant Λ.

It is possible to derive the black hole entropy from holography. As shown below, if we replace the Bekenstein-Hawking entropy (53) with another entropy and we apply the procedure illustrated between Eqs. (88) and (91), the Friedmann equation (91) is modified and extra contributions, which can be seen as holographic dark energy, arise from the non-standard entropy. For example, if we use the Tsallis entropy (54) instead of the Bekenstein-Hawking entropy (53), Eq. (90) is modified to

$$\delta\left(\frac{H}{H_1}\right)^{2(1-\delta)}\dot{H} = -4\pi G\left(\rho + P\right) , \tag{92}$$

where ${H_1}^2 \equiv 4\pi/A_0$. The integration of Eq. (92) yields

$$H^2 = \frac{8\pi G}{3}\left(\rho + \rho_{\mathrm{T}}\right) + \frac{\Lambda}{3} , \quad \rho_{\mathrm{T}} = \frac{3}{8\pi G}\left[H^2 - \frac{\delta}{2-\delta}\, H_1^2\left(\frac{H}{H_1}\right)^{2(2-\delta)}\right] . \tag{93}$$

If we interpret ρ_{T} as the holographic dark energy due to the holographic infrared cutoff $L_{\mathrm{IR,T}}$, $\rho_{\mathrm{T}} = \frac{3C^2}{\kappa^2 {L_{\mathrm{IR,T}}}^2}$, then the holographic infrared cutoff $L_{\mathrm{IR,T}}$ can be identified with

$$\begin{aligned} L_{\mathrm{IR,T}} =& \frac{1}{C\sqrt{H^2 - \frac{\delta}{2-\delta}H_1^2\left(\frac{H}{H_1}\right)^{2(2-\delta)}}} \\ =& \frac{1}{C\sqrt{\left(\frac{\dot{L}_{\mathrm{p}}}{L_{\mathrm{p}}} - \frac{1}{L_{\mathrm{p}}}\right)^2 - \frac{\delta}{2-\delta}\, H_1^2\left(\frac{\frac{\dot{L}_{\mathrm{p}}}{L_{\mathrm{p}}} - \frac{1}{L_{\mathrm{p}}}}{H_1}\right)^{2(2-\delta)}}} \\ =& \frac{1}{C\sqrt{\left(\frac{\dot{L}_{\mathrm{f}}}{L_{\mathrm{f}}} + \frac{1}{L_{\mathrm{f}}}\right)^2 - \frac{\delta}{2-\delta}\, H_1^2\left(\frac{\frac{\dot{L}_{\mathrm{f}}}{L_{\mathrm{f}}} + \frac{1}{L_{\mathrm{f}}}}{H_1}\right)^{2(2-\delta)}}} . \end{aligned} \tag{94}$$

Equivalently, such a FLRW equation can always be rewritten in terms of a generalised cosmological dark fluid. A similar procedure for the Rényi entropy (55) gives

$$\rho_{\mathrm{R}} = \frac{3\pi\alpha}{8G^2}\ln\left(1 + \frac{GH^2}{\pi\alpha}\right) . \tag{95}$$

The three-parameter entropy (70) gives

$$\rho_{\mathrm{G}} = \frac{3}{8\pi G}\left[H^2 - \frac{\pi\alpha}{G\beta\gamma\left(1-\beta\right)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta} {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)\right] , \tag{96}$$

which is expressed in terms of the particle horizon L_p or the future event horizon L_f by

$$\begin{aligned}\rho_\mathrm{G} &= \frac{3}{8\pi G}\left[\left(\frac{\dot{L}_\mathrm{p}}{L_\mathrm{p}}-\frac{1}{L_\mathrm{p}}\right)^2-\frac{\pi\alpha}{G\beta\gamma\left(1-\beta\right)}\left(\frac{G\beta\left(\frac{\dot{L}_\mathrm{p}}{L_\mathrm{p}}-\frac{1}{L_\mathrm{p}}\right)^2}{\pi\alpha}\right)^{2-\beta}\right.\\ &\qquad\left.\times{}_2F_1\left(1-\beta,2-\beta,3-\beta;-\frac{G\beta\left(\frac{\dot{L}_\mathrm{p}}{L_\mathrm{p}}-\frac{1}{L_\mathrm{p}}\right)^2}{\pi\alpha}\right)\right]\\ &= \frac{3}{8\pi G}\left[\left(\frac{\dot{L}_\mathrm{f}}{L_\mathrm{f}}+\frac{1}{L_\mathrm{f}}\right)^2-\frac{\pi\alpha}{G\beta\gamma\left(1-\beta\right)}\left(\frac{G\beta\left(\frac{\dot{L}_\mathrm{f}}{L_\mathrm{f}}+\frac{1}{L_\mathrm{f}}\right)^2}{\pi\alpha}\right)^{2-\beta}\right.\\ &\qquad\left.\times{}_2F_1\left(1-\beta,2-\beta,3-\beta;-\frac{G\beta\left(\frac{\dot{L}_\mathrm{f}}{L_\mathrm{f}}+\frac{1}{L_\mathrm{f}}\right)^2}{\pi\alpha}\right)\right]\,,\end{aligned} \tag{97}$$

where the hypergeometric series terminates and reduces to a polynomial if β is an integer $m \geq 1$. One can define the pressure of the holographic dark energy P_G by means of the covariant conservation law

$$\dot{\rho}_\mathrm{G} + 3H\left(\rho_\mathrm{G}+P_\mathrm{G}\right) = 0\,; \tag{98}$$

the equation of state parameter w_G can then be written as

$$\begin{aligned}w_\mathrm{G} &\equiv \frac{P_\mathrm{G}}{\rho_\mathrm{G}} = -1-\frac{\dot{\rho}_\mathrm{G}}{3H\rho_\mathrm{G}}\\ &= -1-\frac{2}{3}\dot{H}\left[H^2-\frac{\pi\alpha}{G\beta\gamma\left(1-\beta\right)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta}{}_2F_1\left(1-\beta,2-\beta,3-\beta;-\frac{G\beta H^2}{\pi\alpha}\right)\right]^{-1}\\ &\quad\times\left[1-\frac{2-\beta}{\gamma\left(1-\beta\right)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{1-\beta}{}_2F_1\left(1-\beta,2-\beta,3-\beta;-\frac{G\beta H^2}{\pi\alpha}\right)\right.\\ &\qquad\left.+\frac{2-\beta}{\gamma\left(3-\beta\right)}\left(\frac{G\beta H^2}{\pi\alpha}\right)^{2-\beta}{}_2F_1\left(2-\beta,3-\beta,4-\beta;-\frac{G\beta H^2}{\pi\alpha}\right)\right]\,.\end{aligned} \tag{99}$$

When the matter contribution is negligible and the cosmological constant vanishes, the Friedmann equation reads

$$H^2 = \frac{8\pi G}{3}\rho_\mathrm{G} \tag{100}$$

and then Eq. (96) gives

$$ {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right) = 0\,. \tag{101} $$

Therefore, the zeros Z_i of the hypergeometric function ${}_2F_1\left(1-\beta, 2-\beta, 3-\beta; z\right)$ correspond to de Sitter universes with Hubble constant H given by

$$ Z_i = -\frac{G\beta H^2}{\pi\alpha}\,. \tag{102} $$

Then, in spite of the absence of a true cosmological constant Λ, Eq. (102) gives the effective cosmological constant

$$ \Lambda_{\mathrm{eff}} = \frac{3\pi\alpha Z_i}{G\beta}\,. \tag{103} $$

Since H is constant ($\dot H = 0$), if H is given by Eq. (102) the equation of state parameter w_{G} in (99) is almost -1, $w_{\mathrm{G}} \sim -1$. If Λ_{eff} in (103) is large, this effective cosmological constant may describe inflation. On the other hand, if Λ_{eff} is sufficiently small, the effective cosmological constant may describe the accelerated expansion of the present universe. If the effective cosmological constant is slightly larger than the present dark energy, this effective constant could potentially solve the Hubble tension problem.

Let us first consider the case in which Z_i (which we now write as Z_1 for $i = 1$) is sufficiently small. When $\frac{G\beta H^2}{\pi\alpha}$ is small, the hypergeometric function ${}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right)$ is expanded as

$$ \begin{aligned} {}_2F_1\left(1-\beta, 2-\beta, 3-\beta; -\frac{G\beta H^2}{\pi\alpha}\right) &= 1 - \frac{(1-\beta)(2-\beta)}{3-\beta}\frac{G\beta H^2}{\pi\alpha} \\ &\quad + \frac{(1-\beta)(2-\beta)^2}{4-\beta}\left(\frac{G\beta H^2}{\pi\alpha}\right)^2 \\ &\quad + \mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^3\right)\,. \end{aligned} \tag{104} $$

Therefore, if we neglect the terms of order $\mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^2\right)$ in Eq. (104) when H is small, Eqs. (104) and (101) give

$$ Z_1 = -\frac{G\beta H^2}{\pi\alpha} \sim -\frac{(3-\beta)}{(1-\beta)(2-\beta)}\,, \tag{105} $$

that is,

$$H^2 \sim \frac{(3-\beta)\,\pi\alpha}{(1-\beta)\,(2-\beta)\,G\beta} \tag{106}$$

which becomes small when $\beta \lesssim 3$ and the terms of order $\mathcal{O}\left(\left(\frac{G\beta H^2}{\pi\alpha}\right)^2\right)$ in Eq. (104) can be dropped. This conclusion hints at the idea that the solution (106) could explain dark energy in the present universe. We may assume

$$3-\beta \sim \mathcal{O}\left(10^{-2n}\right)\,, \quad \alpha \sim \mathcal{O}\left(10^{-2m}\right)\,, \tag{107}$$

and then Eq. (106) gives

$$H^2 \sim \left(10^{-n-m+28}\,\mathrm{eV}\right)^2\,; \tag{108}$$

therefore, if $n+m=61$, it is $H \sim 10^{-33}\,\mathrm{eV}$, which reproduces the present energy scale of the dark energy. If another zero Z_2 exists with absolute value slightly smaller than Z_1, the effective cosmological constant can potentially solve the Hubble tension problem, *i.e.*, the recent observational tension between the value of the Hubble constant inferred from small redshifts (as in the observations of Type Ia supernova calibrated by Cepheids and that from large redshifts inferred from the cosmic microwave background (CMB). This problem might be solved, or at least alleviated, if there is effectively dark energy just after the CMB was emitted. Our model admitting two zeros $Z_{1,2}$ with $|Z_2|$ slightly larger than $|Z_1|$ might play the role of the effective dark energy just after the CMB.

In general, the hypergeometric function can have several or even an infinite number of zeros. If there are a root of order unity or a large and negative root Z_i of the equation ${}_2F_1\left(1-\beta, 2-\beta, 3-\beta; Z_i\right)=0$, then Eq. (102) can give the large Hubble rate H corresponding to the inflationary epoch. The Hubble rate H and the effective cosmological constant Λ_{eff} are given by Eqs. (102) and (103), respectively. If, for the sake of illustration, we retain the first three terms in Eq. (104), the latter assumes the form

$$1-\frac{(1-\beta)\,(2-\beta)}{3-\beta}\frac{G\beta H^2}{\pi\alpha}+\frac{(1-\beta)\,(2-\beta)^2}{4-\beta}\left(\frac{G\beta H^2}{\pi\alpha}\right)^2=0 \tag{109}$$

with solutions

$$\frac{G\beta H^2}{\pi\alpha}=Z_\pm \equiv -\frac{\frac{(1-\beta)(2-\beta)}{3-\beta} \pm \sqrt{\frac{(1-\beta)^2(2-\beta)^2}{(3-\beta)^2}-4\,\frac{(1-\beta)(2-\beta)^2}{4-\beta}}}{\frac{2(1-\beta)(2-\beta)^2}{4-\beta}}$$

$$= -\frac{4-\beta}{2(2-\beta)(3-\beta)}\left(1 \pm \sqrt{1 - \frac{4(3-\beta)^2}{(4-\beta)(1-\beta)}}\right) . \quad (110)$$

As in Eqs. (107) and (108) we assume $\beta \lesssim 3$, obtaining

$$Z_+ = -\frac{4-\beta}{2(2-\beta)(3-\beta)}, \qquad Z_- = -\frac{3-\beta}{2(1-\beta)(2-\beta)} \quad (111)$$

(here Z_- corresponds to Z_1 in Eq. (105)). Therefore, if one writes α and β as in Eq. (107) and chooses $n+m=61$ as done below Eq. (108), one finds again a Hubble constant H that reproduces the present value of the dark energy scale. If, instead, $\frac{G\beta H^2}{\pi\alpha} = Z_+$, one finds

$$H^2 \sim \left(10^{n-m+28}\,\mathrm{eV}\right)^2 \quad (112)$$

and the choice $n+m=61$ gives

$$H^2 \sim \left(10^{-2m+89}\,\mathrm{eV}\right)^2 . \quad (113)$$

Assuming GUT scale ($\sim 10^{16}\,\mathrm{GeV} = 10^{25}\,\mathrm{eV}$) inflation $H \sim 10^{2\times 25-28}\,\mathrm{eV} = 10^{22}\,\mathrm{eV}$, we obtain $m \sim 33$ or 34. Therefore Z_+ may produce the inflationary epoch of the early universe.

3 CONCLUSION

1. Large number of novel entropies which satisfy the properties of classical entropy and are functions of BH entropy maybe constructed.

2. Odintsov-Paul, PLB831(2022) 137 – Minimum number of parameters of generalised novel entropy is four.

3. The most natural and consistent entropy for BHs is Bekenstein-Hawking entropy.

4. 5 parameters entropy function non-singular during whole universe evolution maybe constructed.

5. entropic FRW eqs coming from novel entropy may describe the whole universe history from inflation till DE!

Search for unique fundamental entropy continues!