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A new approach to the method of nonlinear coherent states

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1. Introduction

- *Coherent states* (CSs) = "objects" (entities) = the state vectors $|z;\lambda>$, $z=|z|\exp(-\mathrm{i}\,\varphi)$
- Expansion in Fock-vector basis $|n;\lambda>$, $n=0,1,\ldots,n_{\max}$, $n_{\max}$ = dim. Hilbert space (λ - real parameter)

$$|z;\lambda>=\frac{1}{\sqrt{N\left(|z|^2;\lambda\right)}}\sum_{n=0}^{n_{\max}}\frac{z^n}{\sqrt{\rho(n;\lambda)}}|n;\lambda> \quad , \quad \rho(n;\lambda)=\begin{cases} n! & \to \text{ linear CSs} = \text{LCSs} \\ n!\,f(n;\lambda) & \to \text{ nonlinear CSs} = \text{NCSs}\end{cases}$$

- $\rho(n;\lambda)$ - *structure constants* (positive), $f(n;\lambda)$ - *nonlinearity (deformation) function* $\Rightarrow$ classification of CSs
- Properties of CSs:[1]

a) Normalization, non-orthogonality: $<z;\lambda|z';\lambda>=\begin{cases} 1, & \text{for } z=z' \\ \neq 0, & \text{for } z\neq z'\end{cases}$

b) Continuity: $z'\to z$, $\lim\limits_{z'\to z}\||z';\lambda>-|z;\lambda>\|^2\to 0$

c) Completeness: $\int d\mu(z;\lambda)|z;\lambda><z;\lambda|=\sum_{n=0}^{n_{\max}}|n;\lambda><n;\lambda|=1$, $d\mu(z;\lambda)=\frac{d\varphi}{2\pi}d\left(|z|^2\right)h\left(|z|^2\right)$

d) temporal stability: CSs must remain coherent over the time

$$\exp\left(-\mathrm{i}\hat{\mathcal{H}}t\right)|z;\lambda>=\exp\left(-\mathrm{i}Ct\right)|z(t);\lambda>,\ z(t)=z\exp\left(-\mathrm{i}\omega t\right),\ C,\omega=\text{const}.$$

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[1] J. R. Klauder, J. Math. Phys. **4**, 8, 1055-1058 (1963).

e) action identity: $<z\,;\lambda\,|\,\hat{\mathcal{H}}\,|\,z'\,;\lambda>=\omega|\,z\,|^2$, $\omega|\,z\,|^2$ is similar to the classical action - variable, if $E_0=0$.

- Historically, there are three types of CSs: ***Barut-Girardello*** (BG-CSs)[2], ***Klauder-Perelomov*** (KP-CSs) [3], and ***Gazeau-Klauder*** (GK-CSs).[4]
- We consider a *dimensionless* Hamiltonian $\hat{\mathcal{H}}$, with *known* eigenvalues $e_{p,q}(n;\,\lambda)$, *p, q* - positive integers

$$\hat{\mathcal{H}}\,|\,n;\lambda>=e_{p,q}(n;\,\lambda)|\,n;\lambda> \quad , \quad e_{p,q}(n;\,\lambda)=n\frac{\prod_{j=1}^{q}(b_j+n)}{\prod_{i=1}^{p}(a_i+n)}$$

- We introduce **two *generalized* hermitic conjugate annihilation / creation operators** $\hat{\mathcal{A}}_-\ /\ \hat{\mathcal{A}}_+$, $\hat{\mathcal{A}}_-=(\hat{\mathcal{A}}_+)^+$
- which generate generalized hypergeometric or nonlinear CSs (NCSs)[5, 6]

$$\hat{\mathcal{A}}_-\,|\,n;\lambda>=\sqrt{e_{p,q}(n)}\,|\,n-1;\lambda>, \qquad \hat{\mathcal{A}}_+\,|\,n;\lambda>=\sqrt{e_{p,q}(n+1)}\,|\,n+1;\lambda>$$

$$\left(\hat{\mathcal{A}}_+\right)^n|\,0;\lambda>=\sqrt{\prod_{m=1}^{n}e_{p,q}(m)}\,|\,n\,;\lambda>=\sqrt{\rho_{p,q}(n;\lambda)}\,|\,n\,;\lambda>\ ,\quad \rho_{p,q}(n;\lambda)\equiv\prod_{m=1}^{n}e_{p,q}(m)\ ,\ \ e_{p,q}(0)=1$$

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[2] A. O. Barut, L. Girardello, Commun. Math. Phys. **21**, 41-55 (1971).
[3] A. M. Perelomov, Commun. Math. Phys. 26 (1972) 222-236; arXiv: math-ph/0203002.
[4] J. P. Gazeau, J. R. Klauder, J. Phys. A: Math. Gen. **32**, 123-132 (1999).
[5] T. Appl, D. H. Schiller, J. Phys. A: Math. Gen., **37**, 7, 2731-2750 (2004).
[6] D. Popov, EJTP **3**, 11, 123 -132 (2006); http://www.ejtp.com/articles/ejtpv3i11p123.pdf

$$|n;\lambda\rangle = \frac{1}{\sqrt{\rho_{p,q}(n;\lambda)}}\left(\hat{\mathcal{A}}_{+}\right)^{n}|0;\lambda\rangle\,, \qquad \langle n;\lambda| = \frac{1}{\sqrt{\rho_{p,q}(n;\lambda)}}\langle 0;\lambda|\left(\hat{\mathcal{A}}_{-}\right)^{n}$$

- We choose the positive **structure functions** as

$$f_{p,q}(n;\lambda) = \frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}\quad , \qquad \rho_{p,q}(n;\lambda) \equiv \prod_{m=1}^{n} e_{p,q}(m) = n!\,\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} = n!\, f_{p,q}(n;\lambda)$$

$(a_i)_n = \Gamma(a_i + n)/\Gamma(a_i)$ - Pochhammer symbols; real numbers $\boldsymbol{a} \equiv a_1, a_2, \ldots, a_p \equiv \{a_i\}_1^p$; $\boldsymbol{b} \equiv b_1, b_2, \ldots, b_q \equiv \{b_j\}_1^q$.

- The ***structure functions*** $\rho_{p,q}(n;\lambda)$ determine the internal structure of NCSs.

- The great advantage of using these operators: ***it is not necessary to know the generators of the quantum group or eigenfunctions***, only to know the energy eigenvalues $E_{p,q}(n) = \hbar\,\omega\, e_{p,q}(n)$.

2. Diagonal ordering operation technique (DOOT)

- We introduced a *new ordering operation*, ***diagonal ordering operation technique*** **(DOOT)**, symbol # #. [7] similar to the IWOP technique (*Integration within Ordered Product*).[8]

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[7] D. Popov, M. Popov, Phys. Scr. **90**, 035101 (2015).
[8] Hong-yi Fan, Commun. Theor. Phys. (Beijing, China) **31**, 2, 285-290 (1999).

- The **fundamental difference**: IWOP applies *only* to the canonical operators, DOOT can be applied to *any pair* of creation / annihilation operators.

- Essential: DOOT ***consists of simply applying the normal ordering rules without taking into account the commutation relation of operators,*** with the **rules:**

a*)** Inside the symbol # # all operators $\hat{\mathcal{A}}_-$ can be moved to the right as if they commuted with $\hat{\mathcal{A}}_+$, to obtain the ***normal ordered operator product $\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-$, or a function $\#\hat{\mathcal{F}}(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)\#$.

b*)**The operators $\hat{\mathcal{A}}_-$ and $\hat{\mathcal{A}}_+$ are treated as ***simple c-numbers, with the properties $\#\hat{\mathcal{A}}_-\hat{\mathcal{A}}_+\# = \#\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\# = \hat{\mathcal{A}}_+\hat{\mathcal{A}}_-$, $\#(\hat{\mathcal{A}}_-)^n(\hat{\mathcal{A}}_+)^n\# = \#(\hat{\mathcal{A}}_+)^n(\hat{\mathcal{A}}_-)^n\# = \#(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)^n\#$.

***c*)** A symbol # # inside another symbol # # can be deleted.

***d*)** If the integration is convergent, we **can integrate or differentiate, with respect to *c*-numbers, according to the usual rules**. The *c*-numbers can be taken out from the symbol # #.

***e*)** The **projector of the vacuum state**, in the DOOT is: $|0;\lambda><0;\lambda| = \#\frac{1}{{}_pF_q(\{a_i\}_1^p;\{b_j\}_1^q;\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)}\#$

- **We *will deal only with the functions which depend on the normal ordered product operators*** $\#\hat{\mathcal{F}}(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)\#$ because the NCSs are expressed in terms of these kinds of functions.

$$\sum_{n=0}^{n_{\max}} |n;\lambda><n;\lambda| = 1, \qquad \Rightarrow \quad |0;\lambda><0;\lambda| = \frac{1}{\#\,{}_pF_q(\boldsymbol{a};\boldsymbol{b};\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)\,\#}$$

- ***Structure functions*** $\rho_{p,q}(n;\lambda)$ are the **moments of a probability distribution** [9]:

$$\rho_{p,q}(n;\lambda) \equiv \int_0^R u^n \rho_{p,q}(u;\lambda)\,du \geq 0 \ , \quad \rho_{p,q}(0;\lambda) = 1 \ , \quad \rho_{p,q}(n;\lambda) < +\infty$$

[9] J. P. Gazeau, J. R. Klauder, J. Phys. A: Math. Gen. **32**, 123-132 (1999).

- Structure functions determine the ***classification of CSs***:

$$|z,\lambda>=\begin{cases}\dfrac{1}{\sqrt{N_L\left(|z|^2;\lambda\right)}}\displaystyle\sum_{n=0}^{n_{\max}}\frac{z^n}{\sqrt{n!}}|n;\lambda>, & \Rightarrow \quad \text{LCSs} \quad \text{linear}\\ \dfrac{1}{\sqrt{N_{NL}\left(|z|^2;\lambda\right)}}\displaystyle\sum_{n=0}^{n_{\max}}\frac{z^n}{\sqrt{n!\,f_{p,q}(n;\lambda)}}|n;\lambda>, & \Rightarrow \quad \text{NLCSs} \quad \text{nonlinear}\end{cases}$$

- Normalization function = **generalized hypergeometric function** (GHf), or **polynomial** (GHp)

$${}_pF_q\left(\boldsymbol{a};\boldsymbol{b};x\right)=\sum_{n=0}^{n_{\max}}\frac{1}{\rho_{p,q}(n;\lambda)}x^n=\sum_{n=0}^{n_{\max}}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{x^n}{n!}\ \Leftrightarrow\ n_{\max}\begin{cases}<\infty, & \Rightarrow \quad GHf\text{ - generalized hypergeometric function}\\ =\infty, & \Rightarrow \quad GHp\text{ - generalized hypergeometric polynomial}\end{cases}$$

- Generally, the lowering / raising operators acts on the CSs for systems with ***infinite energy spectrum*** $\Rightarrow$ definition of BG-CSs:

$$\hat{\mathcal{A}}_-|z;\lambda>=z|z;\lambda>\ ,\qquad <z;\lambda|\hat{\mathcal{A}}_+=z^*<z;\lambda|\qquad ,\qquad <z;\lambda|\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-|z;\lambda>=|z|^2$$

- From DOOT rules, we get

$$<z;\lambda|\mathcal{F}\left(\#\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\#\right)|z;\lambda>=\sum_{j=0}^{n_{\max}}c_j<z;\lambda|\left(\#\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\#\right)^j|z;\lambda>=\sum_{j=0}^{n_{\max}}c_j\left(|z|^2\right)^j=\mathcal{F}\left(|z|^2\right)$$

- ***The practical rule*** for DOOT: in calculating the expected values in the NCSs representation for the ***systems with infinite energy spectrum***, we replace $\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\to|z|^2$.

- The **normalization function** is obtained from $<z\,;\lambda\,|\,z;\lambda>=1$:

$$N\left(|z|^2;\,\lambda\right)=\sum_{n=0}^{n_{\max}}\frac{\left(|z|^2\right)^n}{\rho_{p,q}(n;\lambda)}=\sum_{n=0}^{n_{\max}}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\,\frac{\left(|z|^2\right)^n}{n!}={}_pF_q\left(\boldsymbol{a};\,\boldsymbol{b};|z|^2\right)$$

- Using the properties of $\hat{\mathcal{A}}_-$ and $\hat{\mathcal{A}}_+$, **the NCSs can be written in matrix form**

$$\begin{pmatrix}|z\,;\lambda>\\ \\ <z\,;\lambda\,|\end{pmatrix}=\frac{1}{\sqrt{{}_pF_q(\{a_i\}_1^p;\,\{b_j\}_1^q;|z|^2)}}\begin{pmatrix}{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z\hat{\mathcal{A}}_+)\,|\,0;\lambda>\\ \\ <0\,;\lambda\,|\;{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z^*\hat{\mathcal{A}}_-)\end{pmatrix}$$

- The **projector on the GH-CSs** is, using the DOOT

$$|z\,;\lambda><z\,;\lambda\,|=\frac{1}{{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};|z|^2)}\#\frac{{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z\hat{\mathcal{A}}_+)\,{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z^*\hat{\mathcal{A}}_-)}{{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-)}\#$$

- The **resolution of the unity operator**

$$\int d\mu_{p,q}(z)\,|\,z;\lambda><z;\lambda\,|=1\;,\quad d\mu_{p,q}(z)=\frac{d\varphi}{2\pi}d(|z|^2)h_{p,q}(|z|)\;,\qquad h_{p,q}(|z|)$$ - the weight function

- The **angular integral** involving the hypergeometric functions:

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\#\;{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z\,A_+)\;{}_pF_q(\boldsymbol{a};\,\boldsymbol{b};z^*A_-)\;\#=\sum_{n=0}^{n_{\max}}\frac{\#(A_+A_-)^n\#}{\left[\rho_{p,q}(n;\lambda)\right]^2}\,(|z|^2)^n$$

- Depending on the **convergence radius** R_c, we can have two situations:[10]

$$R_c = \lim_{n\to\infty} \frac{\rho(n)}{\rho(n+1)} = \begin{cases} \infty, & \textit{Stieltjes moment problem } (=SM) \\ <\infty, & \textit{Hausdorff moment problem } (=HM) \end{cases}$$

- For both situations the moment problem is written using the **Heaviside step function**:

$$\int_0^{R_c} d(|z|^2)\, \frac{h_{p,q}(|z|)}{{}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)} (|z|^2)^n H\left(R_c - |z|^2\right) = \rho_{p,q}(n) \qquad , \qquad H\left(R_c - |z|^2\right) = \begin{cases} 0, & |z|^2 > R_c \\ 1, & |z|^2 \le R_c \end{cases}$$

- The **integration measure** is expressed through **Meijer's G-functions**.[11]

$$d\mu_{p,q}(z) = \frac{d\varphi}{2\pi} d(|z|^2) \frac{\prod_{i=1}^{p} \Gamma(a_i)}{\prod_{j=1}^{q} \Gamma(b_j)}\, {}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)\, G^{q+1,0}_{p,q+1}\left(|z|^2 \middle| \begin{matrix} /; & \boldsymbol{a}\textbf{-}\boldsymbol{1} \\ 0, \boldsymbol{b}\textbf{-}\boldsymbol{1}; & / \end{matrix} \right)$$

- The **weight function** must be a **nonoscillatory positive defined function and unique.**

- The **expectation value** of an operator $\hat{\mathcal{A}}$ **in the NCSs representation** is

$$< z;\lambda\,|\,\hat{\mathcal{A}}\,|\,z;\lambda > \equiv < \hat{\mathcal{A}} >_{z;\lambda} = \frac{1}{{}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)} \sum_{n',n=0}^{n_{\max}} \frac{(z^*)^{n'} z^n}{\sqrt{\rho_{p,q}(n')\,\rho_{p,q}(n)}} < n';\lambda\,|\,\hat{\mathcal{A}}\,|\,n;\lambda >$$

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[10] J. R. Klauder, K. A. Penson, J. –M. Sixdeniers, Phys. Rev. A **64**, 013817 (2001).
[11] A. M. Mathai, R. K. Saxena, Lect. Notes Math. Vol. **348** (Springer-Verlag, Berlin, 1973).

- For $\hat{\mathcal{A}} = \hat{\mathcal{N}}^m$ - the integer powers of the **particle number operator**, $\hat{\mathcal{N}} \mid n;\lambda >= n \mid n;\lambda >$

$$< \hat{\mathcal{N}}^m >_{z;\lambda} = \frac{1}{{}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)} \left[|\, z\,|^2 \frac{d}{d(|\, z\,|^2)} \right]^m {}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)$$

- The **statistics** of generalized CSs is revealed by the ***Mandel parameter***: [12]

$$Q_{M,|z|} = \frac{< \hat{\mathcal{N}}^2 >_{z;\lambda} - \left(< \hat{\mathcal{N}} >_{z;\lambda} \right)^2}{< \hat{\mathcal{N}} >_{z;\lambda}} - 1 = |\, z\,| \left[\frac{\left(\frac{d}{d\,|\, z\,|^2} \right)^2 {}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)}{\frac{d}{d\,|\, z\,|^2}\, {}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)} - \frac{\frac{d}{dx}\, {}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)}{{}_pF_q(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2)} \right]$$

- The **Mandel parameter indicates the type of statistics** to which GH-CSs are subject:

$$Q_{M,|z|} = \begin{cases} > 0, & \textit{ordinary (classical) states, called super-Poissonian (bunching states)} \\ = 0, & \textit{canonical states, Poissonian statistics or photon number distribution} \\ < 0, & \textit{non-classical states, said to be sub-Poissonian (anti-bunching)} \end{cases}$$

- The type of statistics to which NCSs obey can also be determined by calculating the **probability distribution** of the transition Fock state – NCSs, versus the Poisson distribution:

$$\mathcal{P}_n\left(|\, z\,|^2 \right) \equiv |< n;\lambda \mid z;\lambda >|^2 = \frac{1}{{}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};|\, z\,|^2\right)} \frac{\left(|\, z\,|^2 \right)^n}{\rho(n;\lambda)} \qquad \Leftrightarrow \qquad \mathcal{P}_n^{\text{Poisson}}\left(|\, z\,|^2 \right) = e^{-|z|^2} \frac{\left(|\, z\,|^2 \right)^n}{n!}$$

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[12] Mandel, L., Wolf, E., *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1994).

$$\mathcal{P}_n\left(|z|^2\right)\begin{cases} < \mathcal{P}_n^{\text{Poisson}}\left(|z|^2\right), & \textit{Sub-Poissonian} \\ = \mathcal{P}_n^{\text{Poisson}}\left(|z|^2\right), & \textit{Poissonian} \\ > \mathcal{P}_n^{\text{Poisson}}\left(|z|^2\right), & \textit{Supra-Poissonian} \end{cases} \quad , \quad \rho_{p,q}(n) = n!\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}\begin{cases} \textit{if} \quad \prod_{j=1}^{q}(b_j)_n > \prod_{i=1}^{p}(a_i)_n & \Rightarrow \textit{Sub-Poissonian} \\ \textit{if} \quad \prod_{j=1}^{q}(b_j)_n = \prod_{i=1}^{p}(a_i)_n & \Rightarrow \textit{Poissonian} \\ \textit{if} \quad \prod_{j=1}^{q}(b_j)_n < \prod_{i=1}^{p}(a_i)_n & \Rightarrow \textit{Supra-Poissonian} \end{cases}$$

3. Mixed (thermal) states

➢ **Mixed state** = superposition of pure states $|...>$) with a certain probability.

➢ The **mixed** (particularly, ***thermal***) **states in the canonical equilibrium with environment**, at temperature $T = (\beta k_B)^{-1}$ are described by a statistical operator **-** ***density operator.*** $Z(\beta)$ **-** ***partition function***

$$\rho(\beta) = \frac{1}{Z(\beta)}\sum_{n=0}^{n_{\max}}\exp\left(-\beta E_n\right)|n;\lambda><n;\lambda| \quad , \quad Z(\beta) = \sum_{n=0}^{n_{\max}}\exp\left(-\beta E_n\right)$$

➢ Using the DOOT rules

$$\rho(\beta) = \frac{1}{Z(\beta)}\# \frac{1}{{}_pF_q\left(\boldsymbol{a};\boldsymbol{b};\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)}\sum_{n=0}^{n_{\max}}\exp\left(-\beta E_n\right)\frac{\left(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)^n}{\rho_{p,q}(n)}\#$$

➢ From dependence of energy $E_n = f(n)$ there are **two categories of systems**:

- with ***linear spectra***, $E_n = \hbar\omega\left(n + e_0\right) = \hbar\omega e(n)$, e.g. harmonic, pseudoharmonic, Pöschl-Teller I oscillators.

- with ***nonlinear spectra***, particularly quadratic, $E_n = \hbar\,\omega\left(e_0 + n + \varepsilon\, n^2\right) = \hbar\,\omega\, e(n)$: Morse, Meixner oscillators.

- We use the **Bose-Einstein distribution function**: $\bar{\boldsymbol{n}} = \dfrac{1}{e^{\beta\,\hbar\,\omega} - 1}$

- For ***linear spectrum*** **(*L*),** with infinite number of bound states, $Z_L(\beta) = \left(\dfrac{\bar{\boldsymbol{n}}}{\bar{\boldsymbol{n}}+1}\right)^{e_0} (\bar{\boldsymbol{n}}+1)$

- The **density operator** of the thermal states for systems with ***linear spectrum*** **(*L*)**, using the DOOT

$$\rho_L(\beta) = \frac{1}{\bar{\boldsymbol{n}}+1} \sum_{n=0}^{n_{\max}} \left(\frac{\bar{\boldsymbol{n}}}{\bar{\boldsymbol{n}}+1}\right)^{n} |\, n\,;\lambda \succ\!\prec n\,;\lambda\,| = \frac{1}{\bar{\boldsymbol{n}}+1} \# \frac{{}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};\, \dfrac{\bar{\boldsymbol{n}}}{\bar{\boldsymbol{n}}+1}\, \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_-\right)}{{}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};\, \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_-\right)} \#$$

- For ***nonlinear (quadratic) spectrum*** we introduced an ***original ansatz*** : we develop the linear part of the energy exponential, in power series, since in most cases $\varepsilon << 1$, (notations:.[13]

$$\exp\left(-\beta\, E_n\right) = e^{-\beta\,\hbar\,\omega\, e_0} \exp\left(-A n\right) \exp\left(-\varepsilon A n^2\right) = e^{-\beta\,\hbar\,\omega\, e_0} \sum_{j=0}^{\infty} \frac{(\varepsilon A)^j}{j\,!} \left(\frac{\partial}{\partial A}\right)^{2j} \exp\left(-A n\right) = e^{-\beta\,\hbar\,\omega\, e_0} \exp\left[\varepsilon\, A\, D_A^2\right] \exp\left(-A n\right)$$

- Notations: $A \equiv \beta\hbar\,\omega$, $B \equiv \beta\hbar\,\omega\,\varepsilon = \varepsilon\, A)$, $\exp\left[\varepsilon\, A\, D_A^2\right] \equiv \exp\left[\varepsilon A \left(\dfrac{\partial}{\partial A}\right)^2\right]$, and the **nonlinear partition function** is

$$Z_{NL}(\beta) = \sum_{n=0}^{n_{\max}} \exp\left(-\beta\, E_n\right) = \left(\frac{\bar{\boldsymbol{n}}}{\bar{\boldsymbol{n}}+1}\right)^{e_0} \exp\left[\varepsilon\, A\, D_A^2\right] \left\{ (\bar{\boldsymbol{n}}+1) \left[1 - \left(\frac{\bar{\boldsymbol{n}}}{\bar{\boldsymbol{n}}+1}\right)^{n_{\max}+1}\right] \right\}$$

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[13] D. Popov, Phys. Lett. A, **316**, 6, 369-381 (2003).

- Using the ***reduced partition function*** $Z_{NL,red}(\beta)$, without the zero energy term.

$$Z_{NL,red}(\beta)\equiv\sum_{n=0}^{n_{\max}}e^{-An-Bn^2}=\exp\left[\varepsilon A D_A^2\right]\sum_{n=0}^{n_{\max}}\exp(-An)=\exp\left[\varepsilon A D_A^2\right]\left\{(\bar{n}+1)\left[1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{n_{\max}+1}\right]\right\} \quad , \quad Z_{NL}(\beta)=\left(\frac{\bar{n}}{\bar{n}+1}\right)^{e_0}Z_{NL,red}(\beta)$$

- ***The linear limit***. For $\varepsilon\to 0$, i.e. $B\to 0$ or $n_{\max}\to\infty$: $\quad \lim_{\varepsilon\to 0}\exp\left[\varepsilon A D_A^2\right]=1 \quad , \quad \lim_{n_{\max}\to\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^{n_{\max}+1}=0$

- Generically, this result can be summarized as follows $\lim\limits_{\substack{\varepsilon\to 0\\ n_{\max}\to\infty}}\mathcal{F}_{NL_{\text{system}}}=\mathcal{F}_{L_{\text{system}}}$

- Generally, using our original ***ansatz***, for **systems with quadratic spectrum, the density operator** is

$$\rho_{NL}(\beta)=\frac{1}{Z_{NL,red}(\beta)}\#\frac{\exp\left[\varepsilon A D_A^2\right]{}_pF_q\left(\boldsymbol{a};\,\boldsymbol{b};e^{-A}\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)}{{}_pF_q\left(\boldsymbol{a};\,\boldsymbol{b};\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)}\#$$

- The mixed thermal states in **quantum optics** are characterized by the ***Q-distribution function*** (***Husimi's function***[14]) = the diagonal elements of the density operator:

$$Q_{NL}\left(|z|^2;\beta\right)\equiv\langle z;\lambda|\,\rho_{NL}(\beta)|\,z;\lambda\rangle=\frac{1}{Z_{NL,red}(\beta)}\frac{\exp\left[\varepsilon A D_A^2\right]{}_pF_q\left(\boldsymbol{a};\,\boldsymbol{b};e^{-A}|z|^2\right)}{{}_pF_q\left(\boldsymbol{a};\,\boldsymbol{b};|z|^2\right)} \quad , \quad 0\le Q\left(|z|^2\right)\le 1\,.$$

- The **diagonal representation of density operator**

$$\rho_{NL}(\beta)=\int d\mu_{p,q}(z)\;P_{p,q,NL}\left(|z|^2;\beta\right)|z\,;\,\lambda\rangle\langle z\,;\,\lambda|$$

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[14] K. Husimi, Proc. Phys. Math. Soc. Jpn. **22**: 264-314 (1940).

- The weighting function $P_{p,q,NL}\left(|z|^2;\beta\right)$ is called ***P-quasi distribution function***, or ***P-representation of density operator***. This is a *quasi*-distribution function, they can be negative in some regions of the complex z plane.
- According to our ansatz, we will write the function *P* in the following manner

$$P_{p,q,NL}\left(|z|^2;\beta\right) \equiv \exp\left[\varepsilon A D_A^2\right] P_{p,q,NL}\left(|z|^2;A,B\right)$$

- **The *final expression for nonlinear P-quasi distribution function***

$$P_{p,q,NL}\left(|z|^2;\beta\right) \equiv \frac{1}{Z_{NL,red}(\beta)} \frac{\exp\left[\varepsilon A D_A^2\right]\left\{ e^A G_{p,q+1}^{q+1,0}\left(e^A |z|^2 \middle| \begin{matrix} /; & \boldsymbol{a}\textbf{-}\boldsymbol{1} \\ 0,\, \boldsymbol{b}\textbf{-}\boldsymbol{1}; & / \end{matrix} \right)\right\}}{G_{p,q+1}^{q+1,0}\left(|z|^2 \middle| \begin{matrix} /; & \boldsymbol{a}\textbf{-}\boldsymbol{1} \\ 0,\, \boldsymbol{b}\textbf{-}\boldsymbol{1}; & / \end{matrix} \right)}$$

$$Q_{NL}\left(|\sigma|^2;\beta\right) = \int d\mu_{p,q}(z)\; P_{p,q,NL}\left(|z|^2;\beta\right) |<z\,;\,\lambda\,|\,\sigma;\,\lambda>|^2$$

- Calculations reveal **an useful new integral in complex space**

$$\int \frac{d^2 z}{\pi} G_{p,q+1}^{q+1,0}\left(|z|^2 \middle| \begin{matrix} /; & \boldsymbol{a}\textbf{-}\boldsymbol{1} \\ 0,\, \boldsymbol{b}\textbf{-}\boldsymbol{1}; & / \end{matrix} \right) {}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};\sigma z^*\right) {}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};\sigma^* z\right) = \Gamma(b/a) e^{-A}\, {}_pF_q\left(\boldsymbol{a};\, \boldsymbol{b};\, e^{-A}|\sigma|^2\right)$$

- The **thermal expectations of operators** (the trace does not depend on the base in which it was calculated)

I. $$<\hat{\mathcal{A}}>_{NL} = \mathrm{Tr}\left(\rho_{NL}\hat{\mathcal{A}}\right) = \int d\mu(z)\; P_{p,q,NL}\left(|z|^2;\beta\right) <z\,;\,\lambda\,|\,\hat{\mathcal{A}}\,|\,z\,;\,\lambda>$$

II. $$<\hat{\mathcal{A}}>_{NL} = \mathrm{Tr}\left(\rho_{NL}\hat{\mathcal{A}}\right) = \frac{1}{Z_{NL,red}(\beta)} \sum_{n=0}^{n_{\max}} e^{-An-Bn^2} <n\,;\lambda\,|\,\hat{\mathcal{A}}\,|\,n\,;\lambda>$$

- E.g., for $\hat{\mathcal{A}} = \hat{\mathcal{N}}^m$ we have $< \hat{\mathcal{N}}^m >_{NL} = \int d\mu(z) P_{p,q,NL}\left(|z|^2; \beta\right) < z\,;\lambda | \hat{\mathcal{N}}^m | z\,;\lambda >$

- The trace can be calculated in **two ways** (the choice being made according to the difficulty of the calculations):

I. **Using the generalized CSs representation**. The expected value $< z\,;\lambda | \hat{\mathcal{N}}^m | z\,;\lambda >$ can also be expressed as follows:

$$< z\,;\lambda | \hat{\mathcal{N}}^m | z\,;\lambda > = \frac{1}{{}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)} \sum_{n=0}^{n_{\max}} n^m \frac{\left(|z|^2\right)^n}{\rho(n;\lambda)} = \frac{1}{{}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)} \left(|z|^2 \frac{\partial}{\partial |z|^2} \right)^m {}_pF_q(\boldsymbol{a};\boldsymbol{b};|z|^2)$$

- Finally, we obtain

$$< \hat{\mathcal{N}}^m >_{NL} = \frac{1}{Z_{NL,red}(\beta)} \exp\left[B D_A^2\right] \left\{ \sum_{n=0}^{n_{\max}} n^m \left(e^{-A}\right)^n \right\} = \frac{1}{Z_{NL,red}(\beta)} \left(e^{-A} \frac{\partial}{\partial e^{-A}} \right)^m Z_{NL,red}(\beta)$$

II. **Using the Fock vectors representation**:

$$< \hat{\mathcal{N}}^m >_{NL} = \frac{1}{Z_{NL,red}(\beta)} \sum_{n=0}^{n_{\max}} e^{-An - Bn^2} < n\,;\lambda | \hat{\mathcal{N}}^m | n\,;\lambda > = \frac{1}{Z_{NL,red}(\beta)} \sum_{n=0}^{n_{\max}} n^m e^{-An - Bn^2}$$

and we will get exactly the result from the previous formula.

- In a previous paper **we defined and introduced** the ***thermal Mandel parameter***, **as the counterpart or thermal analogue of the usual Mandel parameter** [15]

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[15] D. Popov, Phys. Lett. A, **316**, 6, 369-381 (2003).

$$Q_{M,NL}(\beta)\equiv\frac{<\hat{\mathcal{N}}^2>_{NL}-\left(<\hat{\mathcal{N}}>_{NL}\right)^2}{<\hat{\mathcal{N}}>_{NL}}-1=e^{-\beta\hbar\omega}\left[\frac{Z^{(2)}_{NL,red}(\beta)}{Z^{(1)}_{NL,red}(\beta)}-\frac{Z^{(1)}_{NL,red}(\beta)}{Z_{NL,red}(\beta)}\right]>0$$

- Notation for derivatives $Z^{(m)}_{NL,red}(\beta)\equiv\left(e^{-A}\frac{\partial}{\partial e^{-A}}\right)^m Z_{NL,red}(\beta)$.
- The sign was obtained by using the Cauchy-Schwartz inequality: $\left[Z^{(1)}_{NL,red}(\beta)\right]^2\le\left[Z^{(2)}_{NL,red}(\beta)\right]\left[Z_{NL,red}(\beta)\right]$

- The ***Mandel's thermal parameter is positive,*** $\Rightarrow$ ***the corresponding distribution is supra-Poissonian***.
- For systems with ***linear energy spectra*** *with infinite number of bound states*

$$<\hat{\mathcal{N}}^m>_L=\frac{1}{Z_{L,red}(\beta)}\sum_{n=0}^{\infty}\left(e^{-A}\right)^n n^m=\left(1-e^{-A}\right)\left(e^{-A}\frac{\partial}{\partial e^{-A}}\right)^m\frac{1}{1-e^{-A}}$$

- Consequently, **the thermal Mandel parameter is also positive** (supra-Poissonian) $Q_{M,L}(\beta)=\frac{e^{-A}}{1-e^{-A}}=\bar{n}>0$
- *Reminder*: **The Poisson distr**ibution $P_n^{(P)}(\bar{n})$ with the shape parameter $<\hat{\mathcal{N}}>=\bar{n}$ is $P_n^{(P)}(\bar{n})=\exp(-\bar{n})\frac{(\bar{n})^n}{n!}$

- The thermal moments are $<\hat{\mathcal{N}}^m>^{(P)}=\sum_{n=0}^{\infty}n^m P_n^{(P)}(\bar{n})=\exp(-\bar{n})\sum_{n=0}^{\infty}n^m\frac{(\bar{n})^n}{n!}=\exp(-\bar{n})\left(\bar{n}\frac{\partial}{\partial\bar{n}}\right)^m\exp(\bar{n})$

- Consequently, **the Mandel thermal parameter for the Poisson distribution** $P_n^{(P)}(\bar{n})$ **is zero**:

$$Q_M^{(P)}(\beta)=\frac{<\hat{\mathcal{N}}^2>^{(P)}-\left(<\hat{\mathcal{N}}>^{(P)}\right)^2}{<\hat{\mathcal{N}}>^{(P)}}-1=\frac{(\bar{n})^2+\bar{n}-(\bar{n})^2}{\bar{n}}-1=0$$

- The **internal energy *per particle* of a gas of oscillators with a quadratic energy spectrum**.

$$U_{NL}(\beta) = <\hat{\mathcal{H}}>_{NL} = \mathrm{Tr}\left(\rho_{NL}\hat{\mathcal{H}}\right) = -\frac{\partial}{\partial\beta}\ln Z_{NL}(\beta) = -\hbar\,\omega\frac{\partial}{\partial A}\ln Z_{NL}(A)$$

$$U_{NL}(\beta) = \hbar\omega e_0 + \hbar\omega\,\frac{1}{Z_{NL,red}(\beta)}\sum_{n=0}^{n_{\max}} n\,e^{-A(n+\varepsilon n^2)} + \varepsilon\,\hbar\omega\,\frac{1}{Z_{NL,red}(\beta)}\sum_{n=0}^{n_{\max}} n^2 e^{-A(n+\varepsilon n^2)} = \underbrace{\hbar\omega e_0}_{\substack{\text{zero energy}\\ \text{contribution}}} + \underbrace{\hbar\omega <\hat{\mathcal{N}}>_{NL}}_{\substack{\text{harmonic}\\ \text{contribution}}} + \underbrace{\varepsilon\,\hbar\omega <\hat{\mathcal{N}}^2>_{NL}}_{\substack{\text{anharmonic}\\ \text{contribution}}}$$

- At the **harmonic limit** we obtain the average thermal energy of quantum oscillator

$$\lim_{\substack{\varepsilon\to 0\\ n_{\max}\to\infty}} U_{NL}(\beta) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = \frac{1}{2}\hbar\omega\coth\frac{\beta\hbar\omega}{2} = U_{HO-1D}(\beta)$$

4. Duality between the generalized BG- and KP-CSs

- Klauder stated "there are ***a vast number of CSs*** sets, which are distinguished from each other by the presence of different weight factor sets $\rho_{p,q}(n)$" .[16]
- Historically, there are 3 fundamental types of NCSs:
  a) Barut-Girardello coherent states (BG-CSs) = eigenvalues of annihilation operator $\hat{\mathcal{A}}_-$:

$$\hat{\mathcal{A}}_- \,|\, z\,;\lambda>_{BG} = z\,|\, z\,;\lambda>_{BG}$$

b) Klauder-Perelomov coherent states (KP-CSs), by acting of the creation exponential operator (generally, the displacement operator) on the vacuum state:

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[16] J. R. Klauder, *The current state of coherent states,* The 7th ICSSUR Conference, June 2001, Boston, MA, arXiv:quant-ph/0110108

$$|z;\lambda>_{KP}=\frac{1}{\sqrt{N_{KP}\left(|z|^2\right)}}\exp\left(z\hat{\mathcal{A}}_+\right)|0;\lambda>$$

c) Gazeau-Klauder coherent states (GK-CSs) defined in [17] but they can be deduced from BG-CSs, $0\le J\le R\le\infty$ and $-\infty<\gamma<+\infty$: [18]

a.) We define BG-CSs, for the *real* variable $|J>$: $\hat{\mathcal{A}}_-|J>=J|J>$

b.) We develop $|J>$ as a series of the Fock vectors: $|J>=\frac{1}{\sqrt{N_{GK}(J)}}\sum_{n=0}^{n_{\max}}\frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho_{p,q}(n)}}|n>$

c.) We apply the exponential operator $\exp\left(-\mathrm{i}\gamma\hat{\mathcal{H}}\right)=\exp\left(-\mathrm{i}\gamma\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)$ on the state $|J>$

$$|J,\gamma>=\exp\left(-\mathrm{i}\gamma\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)|J>=\frac{1}{\sqrt{N_{GK}(J)}}\sum_{n=0}^{n_{\max}}\frac{\left(\sqrt{J}\right)^n}{\sqrt{\rho_{p,q}(n)}}e^{-\mathrm{i}\gamma e(n)}|n>$$

- All results for BG-CSs can be transcribed for GK-CSs, using the variable equality: $z=\sqrt{J}\exp\left(-\mathrm{i}\gamma\right)$.

$$|J,\gamma>=\frac{1}{\sqrt{N_{GK}(J)}}\sum_{n=0}^{n_{\max}}\frac{\left(\sqrt{J}\right)^n e^{-\mathrm{i}\gamma e_n}}{\sqrt{\rho_{p,q}(n)}}|n>$$

- *Using the DOOT technique, we can establish a series of **dual properties** of NCSs.*

- All NCSs can be written as a superposition (development) of Fock vectors:

$$|z;\lambda>_{\left\{\begin{smallmatrix}BG\\KP\end{smallmatrix}\right\}}=\frac{1}{\sqrt{N_{\left\{\begin{smallmatrix}BG\\KP\end{smallmatrix}\right\}}\left(|z|^2\right)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho_{p,q}^{\left\{\begin{smallmatrix}BG\\KP\end{smallmatrix}\right\}}(n)}}|n;\lambda>=\frac{1}{\sqrt{N_{\left\{\begin{smallmatrix}BG\\KP\end{smallmatrix}\right\}}\left(|z|^2\right)}}N_{\left\{\begin{smallmatrix}BG\\KP\end{smallmatrix}\right\}}\left(z\hat{\mathcal{A}}_+\right)|0;\lambda>$$

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[17] J. P. Gazeau and J. R. Klauder, J. Phys. A: Math. Gen. **32**, 123-132 (1999).
[18] D. Popov, R. Negrea, M. Popov, Chin. Phys. B, **25**, 7, 070301 (2016).

- The two structure constants are connected as

$$\rho_{p,q}^{BG}(n) = n!\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} \quad , \quad \rho_{p,q}^{KP}(n) = n!\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \quad , \quad \rho_{p,q}^{KP}(n) = \frac{(n!)^2}{\rho_{p,q}^{BG}(n)}$$

- This is **a first aspect of duality** $\Rightarrow$ **the normalization functions have inverted coefficients**.

$$N_{BG}\left(|z|^2\right) = {}_pF_q\left(\{a_i\}_1^p ; \{b_j\}_1^q ; |z|^2\right) \quad , \quad N_{KP}\left(|z|^2\right) = {}_qF_p\left(\{b_j\}_1^q ; \{a_i\}_1^p ; |z|^2\right)$$

- The projectors

$$|z;\lambda>_{BG}\ {}_{BG}<z;\lambda| = \frac{1}{\sqrt{N_{BG}\left(|z|^2\right)}} \# \frac{N_{BG}\left(z\hat{\mathcal{A}}_+\right) N_{BG}\left(z^*\hat{\mathcal{A}}_-\right)}{N_{BG}\left(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)} \# \quad , \quad |z;\lambda>_{KP}\ {}_{KP}<z;\lambda| = \frac{1}{\sqrt{N_{KP}\left(|z|^2\right)}} \# \frac{\exp\left(z\hat{\mathcal{A}}_+\right) \exp\left(z^*\hat{\mathcal{A}}_-\right)}{N_{BG}\left(\hat{\mathcal{A}}_+\hat{\mathcal{A}}_-\right)} \#$$

- The integration measures turn out to be

$$d\mu_{BG}(z) = \Gamma(a/b)\frac{d\varphi}{2\pi} d\left(|z|^2\right) G_{p,q+1}^{q+1,0}\left(|z|^2 \left| \begin{matrix} / \ ; & \{a_i - 1\}_1^p \\ 0,\ \{b_j - 1\}_1^q \ ; & / \end{matrix} \right.\right) N_{BG}\left(|z|^2\right)$$

$$d\mu_{GK}(z) = \Gamma(b/a)\frac{d\varphi}{2\pi} d\left(|z|^2\right) G_{q,p+1}^{p+1,0}\left(|z|^2 \left| \begin{matrix} / \ ; & \{b_j - 1\}_1^q \\ 0,\ \{a_i - 1\}_1^p \ ; & / \end{matrix} \right.\right) N_{GK}\left(|z|^2\right)$$

- The consequences of the completeness relations, and using the DOOT rules, *we were able to highlight (discover) the following new integrals*, involving hypergeometric and Meijer's G functions:[19]

$$\int \frac{d^2 z}{\pi} G_{p,q+1}^{q+1,0}\left(|z|^2 \middle| \begin{matrix} / \, ; & \{a_i - 1\}_1^p \\ 0, \{b_j - 1\}_1^q \, ; & / \end{matrix} \right) \# N_{BG}\left(z \hat{\mathcal{A}}_+\right) N_{BG}\left(z^* \hat{\mathcal{A}}_-\right) \# = \Gamma(b/a) \# N_{BG}\left(\hat{\mathcal{A}}_+ \hat{\mathcal{A}}_-\right) \#$$

$$\int \frac{d^2 z}{\pi} G_{q,p+1}^{p+1,0}\left(|z|^2 \middle| \begin{matrix} / \, ; & \{b_j - 1\}_1^q \\ 0, \; \{a_i - 1\}_1^p \, ; & / \end{matrix} \right) \# \exp\left(z \hat{\mathcal{A}}_+\right) \exp\left(z^* \hat{\mathcal{A}}_-\right) \# = \Gamma(a/b) \# N_{BG}\left(\hat{\mathcal{A}}_+ \hat{\mathcal{A}}_-\right) \#$$

- To satisfy the completeness relation, the following integrals must be fulfilled:

for the BG-CSs
$$\int_0^{R_c} d\left(|z|^2\right) G_{p,q+1}^{q+1,0}\left(|z|^2 \middle| \begin{matrix} / \, ; & \{a_i - 1\}_1^p \\ 0, \{b_j - 1\}_1^q \, ; & / \end{matrix} \right) \left(|z|^2\right)^n = \rho_{p,q}^{BG}(n)$$

for the KP-CSs
$$\int_0^{R_c} d\left(|z|^2\right) G_{q,p+1}^{p+1,0}\left(|z|^2 \middle| \begin{matrix} / \, ; & \{b_j - 1\}_1^q \\ 0, \; \{a_i - 1\}_1^p \, ; & / \end{matrix} \right) \left(|z|^2\right)^n = \rho_{p,q}^{KP}(n)$$

- In the DOOT formalism, inside the $\# \;\#$, the operators are treated as simple c-numbers, and can be replaced by some constants, $\Rightarrow$ **these integrals become "purely mathematical".**
- By specifying the indices p and q, as well as the real numbers $\{a_i\}_1^p$ and $\{b_j\}_1^q$, a whole series of particular integrals can be deduced from these general integrals.

[19] D. Popov, R. Negrea, Math. Probl. Eng., Vol. **2022**, Article ID 1284378 (2022).

- This represents a real ***feed-back***: in the elaboration of the theory of CSs, mathematical notions were fully used. Reciprocally, from the properties of CSs, many results were revealed (especially integrals).

Special functions Hypergeometric functions	Definition, properties $\Rightarrow$ $\Leftarrow$ New integrals	***Generalized CSs formalism + DOOT***

- ***Example***. For canonical CSs, if $\hat{\mathcal{A}}_+ \equiv \hat{a}^+$ and $\hat{\mathcal{A}}_- \equiv \hat{a}$, $\Rightarrow$ the integral used in quantum optics:[20]

$$\int \frac{d^2 z}{\pi} \# \exp\left(-|z|^2 + z\hat{a}^+ + z^* \hat{a}\right) \# = \# \exp\left(\hat{a}^+ \hat{a}\right) \# \quad \Leftrightarrow \quad \int \frac{d^2 z}{\pi} e^{A|z|^2 + Bz + Cz^*} = -\frac{1}{A} \exp\left(-\frac{BC}{A}\right), \ \mathrm{Re} A < 0$$

- For thermal states, the differences / dualities between BG-CSs and KP-CSs appear in the expressions of the *Q*- and *P*- distribution functions.

$$Q_L^{(BG)}\left(|z|^2\right) = \frac{1}{\bar{n}+1} \frac{{}_pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; \frac{\bar{n}}{\bar{n}+1}|z|^2\right)}{{}_pF_q\left(\{a_i\}_1^p; \{b_j\}_1^q; |z|^2\right)} \quad , \quad Q_L^{(KP)}\left(|z|^2\right) = \frac{1}{\bar{n}+1} \frac{{}_qF_p\left(\{b_j\}_1^q; \{a_i\}_1^p; \frac{\bar{n}}{\bar{n}+1}|z|^2\right)}{{}_qF_p\left(\{b_j\}_1^q; \{a_i\}_1^p; |z|^2\right)}$$

$$P_L^{(BG)}\left(|z|^2;\beta\right) = \frac{1}{\bar{n}} \frac{G_{p,q+1}^{q+1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2 \left| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right.\right)}{G_{p,q+1}^{q+1,0}\left(|z|^2 \left| \begin{matrix} /; & \{a_i-1\}_1^p \\ 0, \{b_j-1\}_1^q; & / \end{matrix}\right.\right)} \quad , \quad P_L^{(KP)}\left(|z|^2;\beta\right) = \frac{1}{\bar{n}} \frac{G_{q,p+1}^{p+1,0}\left(\frac{\bar{n}+1}{\bar{n}}|z|^2 \left| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0, \{a_i-1\}_1^p; & / \end{matrix}\right.\right)}{G_{q,p+1}^{p+1,0}\left(|z|^2 \left| \begin{matrix} /; & \{b_j-1\}_1^q \\ 0, \{a_i-1\}_1^p; & / \end{matrix}\right.\right)}$$

[20] Hong-yi Fan, Tu-Nan Ruan, Commun. Theor. Phys. **2**, 6, 1563-1574 (1983).

➢ Conclusion: the duality between BG-CSs and KP-CSs , as NCSs consists, in principle, of the mutual interchanges of some entities, as follows:

Entity	***BG-CSs***	***KP-CSs***
Indices interchanges	$p \quad ; \quad q$	$q \quad ; \quad p$
Numbers interchanges	$\boldsymbol{a} \equiv \{a_i\}_1^p \;\; ; \;\; \boldsymbol{b} \equiv \{b_j\}_1^q$	$\boldsymbol{b} \equiv \{b_j\}_1^q \;\; ; \;\; \boldsymbol{a} \equiv \{a_i\}_1^p$
Structure functions	$\rho_{p,q}(n) \equiv \rho_{p,q}^{BG}(n)$	$\rho_{q,p}^{KP}(n) = (n!)^2 \left[\rho_{p,q}^{BG}(n)\right]^{-1}$
Normalization function	$\mathcal{N}_{BG}(A_+A_-) = {}_pF_q(\boldsymbol{a};\boldsymbol{b};\|z\|^2)$	$\mathcal{N}_{KP}(\|z\|^2) = {}_qF_p(\boldsymbol{b};\boldsymbol{a};\|z\|^2)$
Vacuum projector	$\left[\#{}_pF_q(\boldsymbol{a};\boldsymbol{b};A_+A_-)\#\right]^{-1}$	$\left[\#{}_qF_p(\boldsymbol{b};\boldsymbol{a};\tilde{A}_+\tilde{A})\#\right]^{-1}$
Husimi's function	$Q^{(BG)}(\|z\|^2) = f\left[{}_pF_q(\boldsymbol{a};\boldsymbol{b};...)\right]$	$Q^{(KP)}(\|z\|^2) = f\left[{}_qF_p(\boldsymbol{b};\boldsymbol{a};...)\right]$
P-distribution function	$P_{BG}(\|z\|^2;\beta) = f\left[G_{p,\,q+1}^{q+1,0}(...\|\,...)\right]$	$P_{KP}(\|z\|^2;\beta) = f\left[G_{q,\,p+1}^{p+1,0}(...\|\,...)\right]$
Mandel parameter	$Q_{M,\|z\|}^{(BG)} = f\left[\mathcal{N}_{BG}(\|z\|^2)\right]$	$Q_{M,\|z\|}^{(KP)} = f\left[\mathcal{N}_{KP}(\|z\|^2)\right]$
Thermal Mandel parameter	$Q_M(\beta) = f\left[<\hat{\mathcal{N}}^m>\right] = f\left[Z^{(m)}(\beta)\right] > 0$	

5. Concluding remarks

The paper is a *synthesis* regarding a *new approach* of the NCSs which depends essentially on the choice of *structure constants*.

Our important contributions from the paper regarding **NCSs** are:

- ✓ ***Introducing a new procedure for systems with known energy spectrum***, to obtain generalized NCSs, *without needing* to know the *quantum generators*.
- ✓ ***Definition and use of special creation / annihilation operators***, connected with the energy eigenvalues, which generates NCSs.
- ✓ ***Introducing a new technique of normal ordering of creation / annihilation operators*** (DOOT).
- ✓ Rediscovering *known results*, but also obtaining ***some new results*** (e.g. integrals involving hypergeometric and Meijer G functions), using the DOOT.
- ✓ ***Introducing an ansatz applicable to systems with a quadratic energy spectrum***.
- ✓ ***Introducing the thermal Mandel parameter***, as a thermal counterpart of usual Mandel parameter.
- ✓ Revealing in much *more detail the dualism between the BG- and KP- CSs*.

Thank you for your attention ! **Хвала на пажњи !**