Continuation of the nonlinear Schrodinger dynamics to extended space and invariant measures

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Plan

1. Countable system of hyperbolic oscillators and Schrodinger equation with a "weak" nonlinearity admitting blow up effect.

2. Continuation of the flow across the blow up moment.

3. Invariant measures of Hamiltonian flows on a phase space and Koopman unitary representation of groups of Hamiltonian flows.

Evolution Shrodinger equation with nonlinearity

$$i\frac{d}{dt}u(t) = \mathbf{A}(u(t)),\ \ t > 0,$$
$$u(+0) = u_0 \in D \subset H; \qquad \mathbf{A}:\ \ D \to H.$$

We study the blow up effect of nonlinearity for solutions of Cauchy problem. This effect means
the singularity formation on finite time interval,
the unbounded increasing of solution in the norm of the phase space of initial conditions on the boundary of finite time interval.

The above effect for the problem with a "weak"nonlinearity.

$$i\frac{d}{dt}u(t) = \mathbf{\Delta}(\bar{u}(t)),\ \ t > 0,$$
$$u(+0) = u_0 \in D \subset H,$$

where $\mathbf{\Delta}$ is Laplace operator, $\bar{\cdot}$ is the operator of complex conjugation and $D = W_2^2$ is the Sobolev space.

The extension of solution across the singularity formation.

Real Hilbert space with the symplectic structure

Let E be a real separable Hilbert space.

Shift-invariant symplectic form ω on the space E in non-degenerated skew-symmetric bilinear form on E.
There is an ONB $\mathcal{E} = \{e_k\}$ in the space E such that $\omega(e_{2j}, e_n) = \delta_{2j-1,n} \; \forall \; j \in \mathbb{N}$.

$E = P \oplus Q$, $P \tilde{=} Q \tilde{=} l_2$.
$\mathcal{F} = \{f_j\} = \{e_{2j-1}\}$ – ONB in P; $\mathcal{G} = \{g_j\} = \{e_{2j}\}$ – ONB in Q.

$\mathbf{J}$ is linear operator in E associated with the symplectic form

$$\omega(x, y) = (x, \mathbf{J}y)_E$$

$$\mathbf{J}(g_j) = -f_j, \; \mathbf{J}(f_j) = g_j.$$

$$\mathbf{J}^2 = -\mathbf{I}, \; \mathbf{J}^* = -\mathbf{J}.$$

Reification of complex Hilbert space

Let H be a complex Hilbert space.
Bijective mapping $\mathbf{R}: \ H \to E$ is called reification of complex Hilbert space H if
1) There is an ONB $\mathcal{H} = \{h_k\}$ in the space H such that
$\mathbf{R}(u) = p + q, \quad u \in H,$
where $p = \sum_{j=1}^{\infty} f_j \mathrm{Im}(h_j, u) \in P$ and $q = \sum_{j=1}^{\infty} g_j \mathrm{Re}(h_j, u) \in Q$.
2) $\|u\|_H^2 = \|p\|_E^2 + \|q\|_E^2$.

The inverse mapping $\mathbf{C} = (\mathbf{R})^{-1}: \ E \to H$ is called complexification of real Hilbert space E.

$$\mathbf{R}(iu) = \mathbf{J}\mathbf{R}(u) \ \forall \ u \in H,$$

$$(\mathbf{R}(u_1), \mathbf{R}(u_2))_E + i\omega(\mathbf{R}(u_1), \mathbf{R}(u_2)) = (u_1, u_2)_H \ \forall \ u_1, u_2 \in H.$$

$$\tilde{\mathbb{H}} = \frac{1}{2}\sum_{k\in\mathbb{N}} \lambda_k(p_k^2 - q_k^2) = \sum_{k=1}^{\infty} \lambda_k \xi_k \eta_k. \quad (1)$$

Let $z_0 = (p_0, q_0) \in E$ be an initial point of phase trajectory $\mathbf{\Psi}_t(z_0) = z(t, z_0),\ t \in (T_*, T^*)$. Then $z(t, z_0) = (q(t, z_0), p(t, z_0))$ where

$$p_k(t, z_0) = p_{0,k}\,\mathrm{ch}(\lambda_k t) + q_{0,k}\,\mathrm{sh}(\lambda_k t),$$

$$q_k(t, z_0) = q_{0,k}\,\mathrm{ch}(\lambda_k t) + p_{0,k}\,\mathrm{sh}(\lambda_k t);\ \ t \in (T_*, T^*).$$

Lemma 1. $(T_*, T^*) = \mathbb{R}$ iff $\{\lambda_k\} \equiv \Lambda$ is bounded sequence. In this case the phase flow of Hamiltonian system (1)

$$\mathbf{\Psi}_t(q, p) = (\mathrm{ch}(\Lambda t)q + \mathrm{sh}(\Lambda t)p, \mathrm{sh}(\Lambda t)q + \mathrm{ch}(\Lambda t)p),\ \ t \in \mathbb{R}, \quad (2)$$

in the symplectic space (E, ω).

The arising of singularities at finite time

Example: $\lambda_k = -k^2,\ q_{0,k} = 0,\ p_{0,k} = \frac{1}{\mathrm{ch}(k^2)},\ \ k \in \mathbb{N}$.

In this case $(T_*, T^*) = (-1, 1)$,

$\lim\limits_{t\to 1-0} \mathbb{K}(t) = +\infty,\ \lim\limits_{t\to 1-0} \Pi(t) = -\infty,\ \tilde{\mathbb{H}}(p(t), q(t)) = const,$

$\lim\limits_{t\to 1-0} \|p(t)\|_P = \lim\limits_{t\to 1-0} \|q(t)\|_Q = +\infty.$

Let $\mathbf{\Delta}$ be a selfadjoint operator in the space H with the spectrum $\{\lambda_k = -k^2,\ k \in \mathbb{N}\}$. Then

$$\mathbb{H}(u) = -\frac{1}{4}[(\bar{u}, \mathbf{\Delta} u)_H + (u, \mathbf{\Delta}\bar{u})_H] = \frac{1}{4}\int\limits_0^{\pi} [(\frac{\partial \bar{u}}{\partial x})^2 + (\frac{\partial u}{\partial x})^2] dx;$$

$$\tilde{\mathbb{H}}(\mathbf{R}u) = \tilde{\mathbb{H}}(p, q) = \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k [p_k^2 - q_k^2].$$

Thus, hyperbolic oscillator in the space E is described by NSE

$$i\frac{d}{dt}u = \mathbf{\Delta}\bar{u},\ \ t \in (T_*, T^*).$$

An extension of the NSE-dynamics

Now we extend from the space E to the locally convex space of sequences $\mathbb{E} = \mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{\mathbb{N}} \supset E = Q \oplus P$ the symplectic form ω and the symplectic flow $\boldsymbol{\Psi}$.

Locally convex space $\mathbb{E}$ is equipped with the Tikhonov topology, hence the inclusion $E \subset \mathbb{E}$ is dense and continuous.

An extension of symplectic form

A function $\Omega_{\mathbf{J}}: \ \mathbb{E} \times \mathbb{E} \supset D(\Omega_{\mathbf{J}}) \to \mathbb{R}$ is called pseudosymplectic form $\Omega_{\mathbf{J}}$ on the space $\mathbb{E}$ if the following conditions hold.

For every $z \in \mathbb{E}$ there is the linear space $D(z) \subset \mathbb{E}$ such that $\Omega_{\mathbf{J}}(z, \cdot)$ is a linear functional on the space $D(z)$ satisfying conditions:
1) if $y \in D(z)$ then $z \in D(y)$ and $\Omega_J(y, z) = -\Omega_J(z, y)$;
2) if $\Omega_j(z, y) = 0 \ \forall \ y \in D(z)$ then $z = 0$;
3) if $z \in E$ then $D(z) \supset E$ and $\Omega_J(z, y) = \omega_J(z, y) \ \forall \ y \in E$.

For a given $z = (q, p) \in \mathbb{E}$ let us introduce the set $D_{\mathbf{J}}(z) = \{(q', p') \in \mathbb{E} : \ \{q_k p'_k - q'_k p_k\} \in l_1\}$. Then $D_{\mathbf{J}}(z)$ is the linear subspace of LCS $\mathbb{E}$.

If $\Omega_{\mathbf{J}}(z, \cdot): \ D_{\mathbf{J}} \to \mathbb{R}$ by the rule $\Omega_{\mathbf{J}}(z, z') = \sum\limits_{k=1}^{\infty} q_k p'_k - q'_k p_k$ then $\Omega_{\mathbf{J}}$ satisfies the condition 1)-3). Pseudosymplectic form $\Omega_{\mathbf{J}}$ is the continuation of the symplectic form ω to the space $\mathbb{E}$.

Extension of the flow $\mathbf{\Psi}$ onto the space $\mathbb{E}$

Let $\Lambda \in \mathbb{R}^{\mathbb{N}}$ be a sequence of parameters in Hamiltonian (1) of hyperbolic oscillators. Then the formula (2) defines the flow $\mathbf{\Psi}$ in the symplectic space $(\mathbb{E}, \Omega_{\mathbf{J}})$

$$\mathbf{\Psi}_t(p,q) = (\mathrm{ch}(\mathbf{\Lambda}t)p + \mathrm{sh}(\mathbf{\Lambda}t)q,\ \mathrm{ch}(\mathbf{\Lambda}t)q + \mathrm{sh}(\mathbf{\Lambda}t)p),\ t \in \mathbb{R}.$$

This flow is the continuation of the flow in the Euclidean space E across the blow up points T_* and T^*.

The flow $\mathbf{\Psi}$ in the space $(\mathbb{E}, \Omega_{\mathbf{J}})$ preserves the pseudosymplectic form $\Omega_{\mathbf{J}}$.

Now we construct a measure on the space $\mathbb{E}$ which is invariant with respect to the flow $\mathbf{\Psi}$.

Banach limits, Banach measures

On the real line $\mathbb{R}$ there is the unique up to scalar multiplier shift-invariant countable additive measure (Haar measure).
But there are a lot of finitely-additive shift-invariant measures (Banach limits).

There is no Lebesgue measure on an infinitely dimensional Banach space. Since we are interesting in a finitely-additive shift-invariant measure on an infinitely dimensional phase space then we will construct such a measure as the infinite product of shift-invariant finitely-additive measures on a real line.

Let β be a Banach limit on the space $L_\infty(\mathbb{R})$, i.e. $\beta \in L^*_\infty(\mathbb{R})$ such that it is nonnegative shift-invariant and $\beta(\mathbf{1}) = 1$.
If the function ν_β is defined on the σ-algebra $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable subsets by the equality $\nu_\beta(A) = \beta(\chi_A)$ then ν_β is Borel finitely-additive nonnegative shift-invariant measure on real line.

Banach measure of hyperbolic rectangles

Definition 2. A set $\Pi \subset \mathbb{E}$ is called measurable hyperbolic rectangle in the space $\mathbb{E}$ if

$$\Pi = \{(q, p) \in \mathbb{E} :$$

$$(q_i, p_i) = (r_i \operatorname{ch} \varphi_i, r_i \operatorname{sh} \varphi_i), \ (r_i, \varphi_i) \in A_i \times B_i, \ i \in \mathbb{N}\}, \tag{3}$$

where $A_i \times B_i \subset \mathbb{R}_+ \times \mathbb{R}$, A_i, B_i are Borel sets of real line.

Let $\mathcal{K}_\beta(\mathbb{E})$ be the set of measurable hyperbolic rectangles.

Let the function $\lambda_\beta : \ \mathcal{K}_\beta(\mathbb{E}) \to [0, +\infty)$ be defined by the equality

$$\lambda_\beta(\Pi) = \prod_{j=1}^{\infty} \lambda_{2,\beta}(A_j \times B_j), \ \Pi \in \mathcal{K}_\beta(\mathbb{E}), \tag{4}$$

where $\lambda_{2,\beta}(A_j \times B_j) = \nu_\beta(B_j) \int\limits_{A_j} r dr$.

Banach invariant measure of hyperbolic oscillator

Lemma 3. *The function of a set* $\lambda_\beta : \mathcal{K}_\beta(\mathbb{E}) \to [0, +\infty)$ *is additive and invariant with respect to the flow of the hyperbolic oscillators (2).*

Let r_β be a ring generated by the collection of sets $\mathcal{K}_\beta(\mathbb{E})$.

Theorem 4. *Additive function of a set* $\lambda_\beta : \mathcal{K}_\beta(\mathbb{E}) \to [0, +\infty)$ *has the unique additive extension on the ring* r_β*. The completion of the measure* $\lambda_\beta : r_\beta \to [0, +\infty)$ *is the complete measure* $\lambda_\beta : \mathcal{R}_\beta \to [0, +\infty)$*, which is invariant with respect to the Hamiltonian flow of the system of hyperbolic oscillators (2).*

Let $\mathcal{H}_\beta = L_2(\mathbb{E}, \mathcal{R}_\beta, \lambda_\beta, \mathbb{C})$.
In this space we obtain the unitary representation of Hamiltonian flow.

Koopman group of a hyperbolic oscillator

Let $\{\lambda_k\} \equiv \Lambda \in \mathbb{R}^{\mathbb{N}}$. Then

$$\mathbf{\Psi}_t(q,p) = (\mathrm{ch}(\Lambda t)q + \mathrm{sh}(\Lambda t)p, \mathrm{sh}(\Lambda t)q + \mathrm{ch}(\Lambda t)p),\ \ t \in \mathbb{R}.$$

Action-angle coordinates $q_k = r_k\,\mathrm{ch}\,\varphi_k,\ p_k = r_k\,\mathrm{sh}\,\varphi_k,\ \ k \in \mathbb{N}$.
$W\,:\ \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{E}$: $q = r\,\mathrm{ch}\,\varphi,\ p = r\,\mathrm{sh}\,\varphi$

The flow of Hamiltonian system (1) in the action-angle terms:

$$\hat{\mathbf{\Psi}}_t(r,\varphi) = (r, \varphi + \Lambda t), t \in \mathbb{R},\ (r,\varphi) \in \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}^N. \qquad (5)$$

Koopman representation of Hamiltonian flow in the space $\mathcal{H}_\beta$.

$$\mathbf{U}_{\hat{\mathbf{\Psi}}_t}\hat{u}(r,\varphi) = \hat{u}(\hat{\mathbf{\Psi}}_t(r,\varphi)),\ \ \hat{u} \in \mathcal{H}_\beta\ \ t \in \mathbb{R}.$$

Lemma 5. *The Koopman group $\mathbf{U}_{\mathbf{\Psi}}$ of system of hyperbolic oscillators is the unitary group in the space $\mathcal{H}_\beta$ which is strongly continuous iff $\{\lambda_k\}$ is finite sequence.*

Continuity of Koopman group and properties of its generator

Theorem 6. *The Koopman group* $\mathbf{U}_\Psi$ *has the invariant subspace* $\mathcal{H}_\Psi \subset \mathcal{H}_\beta$ *such that the group* $\mathbf{U}_\Psi|_{\mathcal{H}_\Psi}$ *is strongly continuous in the space* $\mathcal{H}_\Psi$*. The generator* $\mathbf{H}_\Psi$ *of the strongly continuous group* $\mathbf{U}_\Psi|_{\mathcal{H}_\Psi}$ *has the continuous set of eigenvalues* $\lambda_{m_1,...,m_N} = m_1\lambda_1 + ... + m_N\lambda_N, \quad N \in \mathbb{N},\ m_1, ..., m_N \in \mathbb{R}$*. Every eigenvalue* $\lambda_{m_1,...,m_N}$ *has the proper eigen subspace*

$$\mathrm{Ker}(\mathbf{H}_\Psi - \lambda_{m_1,...,m_N}\mathbf{I}) \equiv \mathcal{H}_{\vec{m}} = \overline{\mathrm{span}(\prod_{k=1}^{\infty} v_{j_k}(r_k)e^{im_k\varphi_k})}, \qquad (6)$$

where $\vec{m} \in (\mathbb{N} \to \mathbb{R})_0$, $\{v_j\}$ *is an ONB in the space* $L_{2,r}([0,+\infty))$*, and* $\{j_k\}: \ \mathbb{N} \to \mathbb{N}$*.*
The Hilbert space $\oplus_{\vec{m}}\mathcal{H}_{\vec{m}}$ *is the invariant subspace of strong continuity for the Koopman group* $\mathbf{U}_\Psi$*.*

Thank You for Attention

Thank You!