From Fock-Ivanenko covariant derivatives to non-Abelian and Poincare gauge theories

Branislav Sazdović

Institute of Physics,
University of Belgrade, Serbia

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Introduction

- It is well known that
  Gravity is a theory of Poincare gauge symmetries
- We are going to introduce general method to obtain local gauge theory for known global symmery
- First we introduce general matter field $\Psi^A(x)$, where index A contains the set of Lorentz indices (spinors, vectors, tensors, ... any combination of indices).

Local gauge transformations

- For theory invariant under global transformations we will require invariance under corresponding local transformations

$$\Psi'^{A}(x) = R^{A}{}_{B}(x)\Psi^{B}(x) \tag{1}$$

  - $R^{A}{}_{B}(x)$ is representation of the group G

$$R^{A}{}_{B}(x) = \left(e^{-i\Omega(x)}\right)^{A}{}_{B} = \delta^{A}_{B} - i[\Omega(x)]^{A}{}_{B} + \cdots \tag{2}$$

  - $[\Omega(x)]^{A}{}_{B}$ is parameter of transformations

  The infinitesimal variation takes a form

$$\delta\Psi^{A}(x) = R^{A}{}_{B}(x)\Psi^{B}(x) - \Psi^{A}(x) = -i[\Omega(x)]^{A}{}_{B}\Psi^{B}(x)\,. \tag{3}$$

- We need Lagrangian invariant under this transformation. The problem arises with the terms including derivatives. To solve it we must introduce derivative, that transforms in a simple way

Local gauge transformations for $SU(N)$ and Poincare group

- For $SU(N)$ group

$$\Omega^A{}_B(x) = \varepsilon^i(x)(T_i)^A{}_B \tag{4}$$

with parameters $\varepsilon^i(x)$ and corresponding generator $(T_i)^A{}_B$

- For Poincare group

$$\Omega^A{}_B(x) = -\varepsilon^a(x)(P_a)^A{}_B + \frac{i}{2}\omega^{ab}(x)(M_{ab})^A{}_B \tag{5}$$

with parameters $\varepsilon^a(x)$ and $\omega^{ab}(x)$ and corresponding Poincare generator $(P_a)^A{}_B$ and $(M_{ab})^A{}_B$

Parallel transport

- In order to introduce derivative we should subtract values of field $\Psi^A(x)$ in two neighboring points. Except for scalars, we do not know how to do it. So, we will first perform parallel transport of the field $\Psi^A(x)$ from point x^μ to the point y^μ

$$\Psi^A_{\|}(y) = \Pi^A{}_B(y,x)\Psi^B(x)\,, \tag{6}$$

introducing comparator $\Pi^A{}_B(y,x)$.
For infinitesimal separation we have

$$\Pi^A{}_B(x^\mu + \varepsilon n^\mu, x^\mu) = \delta^A_B - i\varepsilon n^\mu (A_\mu(x))^A{}_B + \cdots\,, \tag{7}$$

where $(A_\mu)^A{}_B(x)$ is some general connection

Fock - Ivanenko covariant derivatives

- Now we can define covariant derivative as difference between values of the field Ψ^A at point $x^\mu + \varepsilon n^\mu$ and parallel transport of the field Ψ^A from point x to the point $x^\mu + \varepsilon n^\mu$

$$n^\mu (\mathcal{D}_\mu \Psi)^A = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi^A (x^\mu + \varepsilon n^\mu) - \Psi^A_\parallel (x^\mu + \varepsilon n^\mu) \right]$$
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\Psi^A (x^\mu + \varepsilon n^\mu) - \Pi^A{}_B (x^\mu + \varepsilon n^\mu, x^\mu) \Psi^B (x) \right]$$
$$= n^\mu \left[\partial_\mu \Psi^A (x) + i (A_\mu (x))^A{}_B \Psi^B (x) \right] . \qquad (8)$$

Consequently, Fock - Ivanenko covariant derivatives with respect to general connection $(A_\mu(x))^A{}_B$ has a form

$$(\mathcal{D}_\mu)^A{}_B = \delta^A_B \partial_\mu + i [A_\mu (x)]^A{}_B . \qquad (9)$$

Transformation of comparator $\Pi^A{}_B(y,x)$

- We already had expression for parallel transport of the field $\Psi^A(x)$ from point x^μ to the point y^μ

$$\Psi^A_{\parallel}(y) = \Pi^A{}_B(y,x)\Psi^B(x) \tag{10}$$

Since the transported field $\Psi^A_{\parallel}(y)$, transforms as

$$\Psi'^A_{\parallel}(y) = R^A{}_B(y)\Psi^A_{\parallel}(y)\,, \tag{11}$$

the comparator $\Pi^A{}_B(y,x)$ has transformation

$$\Pi'^A{}_B(y,x) = R^A{}_C(y)\Pi^C{}_D(y,x)R^{\dagger D}{}_B(x)\,. \tag{12}$$

Transformation lows for covariant derivatives

- The transformation low for covariant derivatives follows from definition of covariant derivatives and transformation of the fields

$$(\mathcal{D}'_\mu)^A{}_B = R^A{}_C(x)(\mathcal{D}_\mu)^C{}_D R^{\dagger D}{}_B(x)\,. \tag{13}$$

- So, we obtain covariant derivative, that transforms in a simple way

Covariant derivative of product

- Covariant derivative of product of the fields

$$
\begin{aligned}
& n^{\mu}[\mathcal{D}_{\mu}(\Psi_1\Psi_2)]^{AB} = \\
& \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[(\Psi_1^A\Psi_2^B)(x^{\mu}+\varepsilon n^{\mu}) - (\Psi_{1\|}^A\Psi_{2\|}^B)(x^{\mu}+\varepsilon n^{\mu})\right] \\
& = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[(\Psi_1^A\Psi_2^B)(x^{\mu}+\varepsilon n^{\mu})\right. \\
& \left. -\Pi^A{}_C(x^{\mu}+\varepsilon n^{\mu},x^{\mu})\Psi_1^C(x)\Pi^B{}_D(x^{\mu}+\varepsilon n^{\mu},x^{\mu})\Psi_2^D(x)\right] \\
& = n^{\mu}\left[\partial_{\mu}(\Psi_1^A\Psi_2^B) + i(A_{\mu})^A{}_C\Psi_1^C\Psi_2^B + i\Psi_1^A(A_{\mu})^B{}_D\Psi_2^D\right], \quad (14)
\end{aligned}
$$

Universality of covariant derivatives

- So, we obtain

$$[\mathcal{D}_\mu(\Psi_1\Psi_2)]^{AB} = (\mathcal{D}_\mu\Psi_1)^A\Psi_2^B + \Psi_1^A\mathcal{D}_\mu\Psi_2^B\,. \tag{15}$$

- Therefore, the Leibniz rule valid

From general connection and general covariant derivatives to standard ones

- As well as gauge parameter $\Omega^A{}_B(x)$ we can expand general connection $[A_\mu(x)]^A{}_B$ in terms of group generators
- For example if generators are spin part of Lorentz subgroup $(S_{ab})^A{}_B$ the corresponding coefficients $A^{ab}_\mu(x)$ are spin connection

$$[A_\mu(x)]^A{}_B = \frac{1}{2}\omega^{ab}_\mu(x)(S_{ab})^A{}_B \tag{16}$$

General connection for vectors and spinors

- In particular case for vector fields we have

$$[A_\mu(x)]^c{}_d = \frac{1}{2}\omega_\mu^{ab}(x)(S_{ab})^c{}_d \,, \tag{17}$$

$$(S_{ab})^c{}_d = i\left(\delta_a^c \eta_{bd} - \delta_b^c \eta_{ad}\right) . \tag{18}$$

- and for spinor fields

$$(A_\mu(x))^\alpha{}_\beta = \frac{1}{2}\omega_\mu^{bc}(x)(S_{bc})^\alpha{}_\beta \,, \tag{19}$$

$$(S_{bc})^\alpha{}_\beta = \frac{i}{4}[\gamma_b, \gamma_c] \tag{20}$$

- where $\omega_\mu^{bc}(x)$ is spin connection

Standard covariant derivatives for vectors

- Application of Fock - Ivanenko covariant derivatives on the field $\Psi^A(x)$ produces standard covariant derivatives of this field
- For vector fields we have

$$[A_\mu(x)]^c{}_d = \frac{i}{2}\omega_\mu^{ab}(x)\Big(\delta_a^c\eta_{bd} - \delta_b^c\eta_{ad}\Big) = i(\omega_\mu)^c{}_d(x) \quad (21)$$

which produces

$$(\mathcal{D}_\mu)^c{}_d = \delta_d^c\partial_\mu + i[A_\mu(x)]^c{}_d = \delta_d^c\partial_\mu - (\omega_\mu)^c{}_d(x) \quad (22)$$

or

$$(D_\mu V)^c = (\mathcal{D}_\mu)^c{}_d V^d = \partial_\mu V^c - (\omega_\mu)^c{}_d(x) V^d \quad (23)$$

which is standard expression for covariant derivatives on vector

Universality of covariant derivatives - Example

- Choosing spinor fields $\Psi^A \to \psi^\alpha$ we have

$$[\mathcal{D}_\mu(\psi_1\psi_2)]^{\alpha\beta} = (\mathcal{D}_\mu\psi_1)^\alpha\psi_2^\beta + \psi_1^\alpha(\mathcal{D}_\mu\psi_2)^\beta \,. \tag{24}$$

- Introducing vector field

$$V^a = \bar{\psi}\gamma^a\psi = \bar{\psi}^\alpha\gamma^a_{\alpha\beta}\psi^\beta \,, \tag{25}$$

we obtain

$$(\mathcal{D}_\mu V)^a = \gamma^a_{\alpha\beta}\left[(\mathcal{D}_\mu\bar{\psi})^\alpha\psi^\beta + \bar{\psi}^\alpha(\mathcal{D}_\mu\psi)^\beta\right] \tag{26}$$

This is essence of the publications
V. A. Fock, D. D. Ivanenko, *Compt. Rend.* , **188** (1929) 1470
V. A. Fock, *Zs. f. Phys.*, **57** (1929) 261
Following some other papers, I used the name Fock - Ivanenko covariant derivatives

Infinitesimal construction of field strength – Commutator of covariant derivatives $\mathcal{D}_\mu$

- Commutator of covariant derivatives

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]^A{}_B \Psi^B = (\mathcal{D}_\mu)^A{}_B (\mathcal{D}_\nu \Psi)^B - (\mathcal{D}_\nu)^A{}_B (\mathcal{D}_\mu \Psi)^B \qquad (27)$$

so that

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]^A{}_B \Psi^B = \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\Big)^A{}_B \qquad (28)$$

- If we introduce general field strength

$$(\mathcal{F}_{\mu\nu}(A))^A{}_B = \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\Big)^A{}_B \,, \qquad (29)$$

we have

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]^A{}_B = i(\mathcal{F}_{\mu\nu}(A))^A{}_B \,. \qquad (30)$$

Integral construction of field strength – "Closed path" comparison

- We can measures difference between initial value of field Ψ^A and its final value $\Psi^A_{\|}(x)$ (after parallel transport) along some infinitesimal closed curve $\mathcal{C}$. The difference is proportional to the the area $\varepsilon^2 n_1^\mu n_2^\nu$ whose boundary is closed curve $\mathcal{C}$. So, we have

$$i n_1^\mu n_2^\nu (\mathcal{C}_{\mu\nu}\Psi)^A =$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\Psi^A(x^\mu) - \Psi^A_{\|}(x^\mu)\right] = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\Psi^A(x^\mu) - \Pi^A{}_B(x^\mu)\Psi^B(x)\right].$$

"Closed path" comparison does not involve any derivatives on $\Psi^A(x^\mu)$

- Let us show that the "closed path" comparison acting on a field $\Psi^A(x^\mu)$ does not involve any derivatives on $\Psi^A(x^\mu)$. In fact, since the difference is at the same point and scalar field does not change after parallel transport, for any scalar field $\varphi(x)$ we have

$$in_1^\mu n_2^\nu [\mathcal{C}_{\mu\nu}(\varphi\,\Psi)]^A = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left[\varphi(x)\,\Psi^A(x^\mu) - \varphi(x)\,\Psi^A_{\parallel}(x^\mu)\right]$$

$$= \varphi(x)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left[\Psi^A(x^\mu) - \Psi^A_{\parallel}(x^\mu)\right] = \varphi(x) in_1^\mu n_2^\nu (\mathcal{C}_{\mu\nu}\Psi)^A \,. \quad (32)$$

This means that $\mathcal{C}_{\mu\nu}$ cannot depend on derivatives of $\Psi^A(x^\mu)$, because if it did it would also have to depend on derivatives of $\varphi(x)$.
Therefore, we introduced name comparison instead derivative.

"Closed path" comparison – calculation

- Comparator for parallel transport along infinitesimal closed curve is

$$\Pi^A{}_B(x) = \delta^A_B - i\varepsilon^2 n_1^\mu n_2^\nu (\mathcal{F}_{\mu\nu})^A{}_B\,. \tag{33}$$

- So we have

$$\begin{aligned} i n_1^\mu n_2^\nu (\mathcal{C}_{\mu\nu}\Psi)^A &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\Psi^A(x^\mu) - \Pi^A{}_B(x^\mu)\Psi^B(x)\right] \\ &= i n_1^\mu n_2^\nu (\mathcal{F}_{\mu\nu})^A{}_B \Psi^B(x)\,, \end{aligned} \tag{34}$$

which produces

$$(\mathcal{C}_{\mu\nu})^A{}_B = (\mathcal{F}_{\mu\nu})^A{}_B\,. \tag{35}$$

Field strength for Poincare group produces of torsion and curvature

- Field strength

$$(\mathcal{F}_{\mu\nu}(A))^A{}_B = \Big(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\Big)^A{}_B \,, \qquad (36)$$

- Let us first derive commutator $[A_\mu(x), A_\nu(x)]$ Lie algebra of Poincare group

$$[A_\mu(x), A_\nu(x)]^A{}_B = -\frac{1}{2} B^a_\mu(x)\omega^{cd}_\nu(x)[P_a, M_{cd}]^A{}_B$$
$$-\frac{1}{2}\omega^{ab}_\mu(x)B^c_\nu(x)[M_{ab}, P_c]^A{}_B + \frac{1}{4}\omega^{ab}_\mu(x)\omega^{cd}_\nu(x)[M_{ab}, M_{cd}]^A{}_B \,, (37)$$

or

$$[A_\mu(x), A_\nu(x)]^A{}_B = i\Big[\omega^a_{\mu b}(x)B^b_\nu(x) - \omega^a_{\nu b}(x)B^b_\mu(x)\Big](P_a)^A{}_B$$
$$+\frac{i}{2}\Big[\omega^a_{\mu c}(x)\omega^{cb}_\nu(x) - \omega^a_{\nu c}(x)\omega^{cb}_\mu(x)\Big](M_{ab})^A{}_B \,. (38)$$

Torsion and curvature

- We can write field strength for Poincare group linear in generators

$$(\mathcal{F}_{\mu\nu}(A))^{A}{}_{B} = T^{a}_{\mu\nu}(P_a)^{A}{}_{B} - \frac{1}{2}R^{ab}{}_{\mu\nu}(\omega)(M_{ab})^{A}{}_{B} \qquad (40)$$

- curvature

$$R^{ab}{}_{\mu\nu}(\omega) = \partial_\mu \omega^{ab}_{\nu} - \partial_\nu \omega^{ab}_{\mu} + \omega^{ac}_{\mu}\omega_{\nu c}{}^{b} - \omega^{ac}_{\nu}\omega_{\mu c}{}^{b}\,, \qquad (41)$$

- torsion

$$T^{a}_{\mu\nu} = \partial_\mu B^{a}_{\nu} + \omega^{a}_{\mu}{}_{b}B^{b}_{\nu} - \partial_\nu B^{a}_{\mu} - \omega^{a}_{\nu}{}_{b}B^{b}_{\mu} = D_\mu B^{a}_{\nu} - D_\nu B^{a}_{\mu}\,, \qquad (42)$$

Note that curvature $R^{ab}{}_{\mu\nu}(\omega)$ and torsion $T^{a}_{\mu\nu}$ do not depend on representation

Transformations of torsion and curvature

$$\delta(\mathcal{F}_{\mu\nu}(A))^A{}_B = -i[\Omega, (\mathcal{F}_{\mu\nu}(A))]^A{}_B \,. \tag{43}$$

For Poincare group

$$\delta_\omega T^a{}_{\mu\nu} = -\omega^a{}_c T^c{}_{\mu\nu} \,, \qquad \delta_\varepsilon T^a{}_{\mu\nu} = R^{ab}{}_{\mu\nu}(\omega)\varepsilon_b \,, \tag{44}$$

and

$$\delta_\omega R^{ab}{}_{\mu\nu}(\omega) = \omega^a{}_c(x) R^{cb}{}_{\mu\nu}(\omega) + \omega^b{}_c(x) R^{ac}{}_{\mu\nu}(\omega) \,,$$
$$\delta_\varepsilon R^{ab}{}_{\mu\nu}(\omega) = 0 \,. \tag{45}$$

Tetrad field

- The field $\Psi^A[x^a(x^\mu)]$ is function of local coordinates x^a which depends on space-time coordinates x^μ. Since x^a is vector in flat space we can apply definition of covariant derivative to the vector x^a instead to Ψ^A.
- If we introduce

$$dx^\mu \equiv \varepsilon n^\mu \, ,$$
$$dx^a \equiv x^a(x^\mu + \varepsilon n^\mu) - x^a_{\parallel}(x^\mu + \varepsilon n^\mu) \tag{46}$$

then we can define tetrad field $e^a{}_\mu(x)$ as

$$dx^a = dx^\mu (\mathcal{D}_\mu x)^a = dx^\mu e^a{}_\mu(x) \tag{47}$$

- then

$$e^a{}_\mu(x) = (\mathcal{D}_\mu)^a{}_b x^b = \partial_\mu x^a + i(A_\mu(x))^a{}_b x^b \, . \tag{48}$$

Tetrad field - 2

- Using expression for the Fock - Ivanenko covariant derivatives with respect to both $\omega_\mu^{ab}(x)$ and $B_\mu^a(x)$ we obtain

$$e^a{}_\mu(x) = \left[\delta_b^a \partial_\mu + iB_\mu^c(x)(P_c)^a{}_b - \frac{i}{2}\omega_\mu^{cd}(x)(M_{cd})^a{}_b\right]x^b \,. \quad (49)$$

Using

$$(P_c)^a{}_b = \delta_b^a i\partial_c \,,$$

$$(M_{cd})^a{}_b = i\left(\delta_c^a\,\eta_{db} - \delta_d^a\,\eta_{bc}\right) + i\delta_b^a(x_c\partial_d - x_d\partial_c)\,, \quad (50)$$

we have

$$(M_{cd})^a{}_b x^b = 0\,. \quad (51)$$

- Consequently

$$e^a{}_\mu(x) = \partial_\mu x^a - B_\mu^a(x)\,. \quad (52)$$

Metric tensor

- Now the expression for interval takes a form

$$ds^2 = \eta_{ab} dx^a dx^b = \eta_{ab} e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \,, \quad (53)$$

where we introduced the metric tensor

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,. \quad (54)$$

In terms of gauge field $B^a_\mu(x)$ the metric tensor has a form

$$g_{\mu\nu} = \eta_{ab}(\partial_\mu x^a - B^a_\mu)(\partial_\nu x^b - B^b_\nu) \,. \quad (55)$$

Gauge invariant Lagrangians

- In the Yang-Mills case it is not possible to construct Lagrangian linear in field strength, because $Tr(T^i) = 0$. So, the nontrivial one is bilinear in field strength
- For local Poincare group there is the other possibility to construct invariants which produces scalar curvature

$$R^{ab}{}_{\mu\nu}(\omega)(S_{ab})^c{}_d \, e^\mu{}_c \, e^{\nu d} = 2iR \tag{56}$$

So, we can chose Lagrangian

$$\mathcal{L}_1 = \frac{1}{2i} R^{ab}{}_{\mu\nu}(\omega)(S_{ab})^c{}_d \, e^\mu{}_c \, e^{\nu d} = R \, . \tag{57}$$

Gauge invariant Lagrangians - 1

- We can find bilinear form with local Poincare group

$$\mathcal{L}_2 = Tr\Big(R^{ab}{}_{\mu\nu} S_{ab}\Big)^2 = 4R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \tag{58}$$

- To construct a theory of matter field interacting with gravitational field, we can take standard Lagrangian for matter field and replace ordinary derivative with covariant one

$$\partial_\mu \to D_\mu = \partial_\mu - \frac{i}{2}\,\omega_\mu^{ab}\, S_{ab}\,. \tag{59}$$