Spaces of negative dimensions in black hole thermodynamics and quantum cosmology

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Outlook

- **Introduction**
- **BH/BG correspondence**
- **Negative dimensions**
- **Cosmological constant**
- **Quantum gravity: dimension change transitions**

- **Multidimensional spaces are widely used in mathematics and physics.**
- **This talk will discuss the emergence of negative dimensions in black hole thermodynamics, quantum cosmology and theory of Bose gas.**

Aref'eva, I.V., "*Violation of the Third Law of Thermodynamics by Black Holes, Riemann Zeta Function and Bose Gas in Negative Dimensions***," 2304.04695**

Introduction. Thermodynamics of black holes

- **Schwarzschild black hole**

$$ds^2 = -f(r)dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2_{D-2},$$

$$f(r_h) = 0 \qquad \textbf{horizon}$$

$$T = \frac{1}{4\pi} f'(r)\Big|_{r=r_h} \ \textbf{Hawking temperature, Bogolubov tr.}$$

$$\frac{Area}{4} \ \textbf{entropy}$$

Introduction. Third Law of Thermodynamics

- **In the Planck formulation:**

Entropy $S \to 0$ as $T \to 0$ $(\beta = \frac{1}{T} \to \infty)$

Introduction. Violation of Third Law in BH Thermodynamics

- **Schwarzschild black hole**
  - **Hawking temperature** $T = \frac{1}{8\pi M}$
  - **Bekenstein-Hawking entropy** $S = \frac{1}{16\pi T^2} \to \infty \quad as \quad T \to 0$

- **Black holes violate the third law of thermodynamics (*in Planck formulation*).**

- We are looking for quantum statistical models with black hole thermodynamic behavior.

- It will be shown that a Schwarzschild black hole in D=4 spacetime dimensions corresponds to a Bose gas in space with d=-4 negative spatial dimensions.

- The Riemann zeta function is used to determine the entropy of a Bose gas in negative dimensions.

Introduction. Microscopic origin of BH entropy

- **The problem of the microscopic origin of the Bekenstein-Hawking entropy of a black hole has attracted a lot of attention over the past 30 years**
  - **Wheeler** considered of the BH interior as "bag of gold"
  - **Strominger and Vafa, 96'** D-0 branes interpretation

    ,
  - **'t Hooft 84'** proposed to relate BH entropy with the entropy of thermally excited quantum fields in the vicinity of the horizon.
  - Matrix models corresponding to BH in spacetime with topology $AdS_2 \times S^8$, **Maldacena'23**

To summarize Introduction

- **Schwarzschild BHs violate 3-d law of thermodynamics.**

  Indeed, Schwarzschild BH entropies in D-dim $S \to \infty$ rather than zero when $T \to 0$.

- **We search for quantum statistical models with such exotic thermodynamic behaviour.**

Bose gas in negative dimensions

Bose Gas

- $\mathcal{H}$ Hilbert space, $L_2(\mathcal{M})$, $\quad \mathcal{M} \subset \mathbb{R}^d$, $\mathcal{F} = \bigoplus \mathcal{H}^{\otimes n}$

H is an operator in $\mathcal{H}$

$$F = -\frac{1}{\beta} \lim_{|\mathcal{M}| \to \infty} \frac{1}{|\mathcal{M}|} \log \operatorname{Tr} e^{\beta(\mu \hat{N} - \hat{H})}$$

$$F_{BG} = \frac{\Omega_{d-1}}{\beta} \left(\frac{L}{2\pi} \right)^d \int_0^\infty \ln \left(1 - e^{-\beta \, \varepsilon(k)} \right) \, k^{d-1} dk$$

- Free energy F, $S = \beta^2 \partial_\beta F$
- $\varepsilon(k) = \lambda\, k^2$

$$F_{BG} = \frac{\Omega_{d-1}}{\beta} \int_0^\infty \ln\left(1 - e^{-\beta\, \varepsilon(k)}\right) k^{d-1} dk.$$

$$F_{BG} = -\,\pi^{d/2} \left(\frac{1}{\beta}\right)^{\frac{d}{2}+1} \left(\frac{1}{\lambda}\right)^{\frac{d}{2}} \zeta\left(\frac{d}{2} + 1\right).$$

- $\varepsilon(k) = \lambda\, k^\alpha,\ \lambda > 0,\ \alpha > 0.$
- **Bose gas free energy**

$$F_{BG} \ = \ \frac{\Omega_{d-1}}{\beta} \int_0^\infty \ln\left(1 - e^{-\lambda\,\beta\ k^\alpha}\right)\, k^{d-1} dk$$

Making the change of variables $\lambda\beta k^\alpha = x$

$$F_{BG} = -\frac{2\pi^{d/2}}{d\Gamma(d/2)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right).$$

D=4 Schwarzschild BH vs Bose Gas.

- **For** $\varepsilon(k) = \lambda\, k^2$ **equalizing** $F_{BG}(\beta) = F_{BH}(\beta)$

$$-\frac{\pi^{d/2}}{\beta^{\frac{d}{2}+1}\lambda^{\frac{d}{2}}}\,\zeta\left(\frac{d}{2}+1\right) = \frac{\beta}{16\pi} \qquad (*)$$

- **To fulfill (*) we have to assume**

$$\mathbf{d{=}\text{-}\,4}, \qquad \lambda^2 = -\,\frac{\pi}{16\,\zeta(-1)}.$$

- **Taking into account that** $\zeta(-1) = -1/12$**, we get**

$$\lambda \;=\; \sqrt{\frac{3\pi}{4}},$$

BH/BG correspondence

- **Theorem.** *The thermodynamics of D=4 Schwarzschild BH is equivalent to the thermodynamics $d = -4$ Bose Gas, i.e.*

$$F_{BH}(D, \beta) = F_{BG}(d, \beta)$$

- D**-dimensional Schwarzschild black hole,** $D \geq 4$,

$$ds^2 = -\left(1 - \frac{{r_h}^{D-3}}{r^{D-3}}\right) dt^2 + \frac{dr^2}{1 - \frac{{r_h}^{D-3}}{r^{D-3}}} + r^2 d\omega_{D-2}^2,$$

- **Hawking temperature** $T = 1/\beta = \frac{D-3}{4\pi r_h}$
  r_h **is the radius of the horizon.**
- **The entropy and the free energy are**

$$S = \frac{\Omega_{D-2}}{4}\left(\frac{D-3}{4\pi}\,\frac{1}{T}\right)^{D-2} ;\; F = \frac{(D-3)^{D-3}\beta^{D-3}\,\Omega_{D-2}}{4(4\pi)^{D-2}}$$

$S \to \infty$, **when** $T \to 0$, **a violation of the 3-d law**

- **Equalizing:** $F_{BG}(d,\beta) = F_{BH}(D,\beta)$ **we get**

$$-\frac{\pi^{d/2}}{\Gamma(d/2+1)}\left(\frac{1}{\beta}\right)^{\frac{d}{\alpha}+1}\left(\frac{1}{\lambda}\right)^{\frac{d}{\alpha}}\Gamma\left(\frac{d}{\alpha}+1\right)\zeta\left(\frac{d}{\alpha}+1\right)$$

$$= \frac{(D-3)^{D-3}}{4(4\pi)^{D-2}}\beta^{D-3}\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$$

- **powers of** β**:**

$$d = -(D-2)\,\alpha$$

- **Theorem.** [BH/BG-correspondence]
  The thermodynamics of D dimensional Schwarzschild BH is equivalent to the thermodynamics d Bose Gas, i.e.

$$F_{BH}(D,\beta) = F_{BG}(d,\beta)$$

if the D, d *and* α *are related as presented in the Table*

D	d	α
$D = 4k+1, \quad k = 1,2,3...$	$d = (4k-1)\lvert\alpha\rvert$	$\alpha = -1, -2, \ldots$
$D = 4k+1, \quad k = 1,2,3...$	$d = -(4k-1)\alpha$	$\frac{4r}{4k-1} < \alpha < \frac{2(2r+1)}{4k-1}, \quad r = 0,1,2,\ldots$
$D = 4k+3, \quad k = 1,2,3...$	$d = -(4k+1)\alpha$	$\frac{2(2r+1)}{4k+1} < \alpha < \frac{4(r+1)}{4k+1}, \quad r = 0,1,2...$
$D = 2k, \quad k = 2,3,4...$	$d = -2(k-1)\alpha$	$\alpha = \frac{k+2n}{k-1}, \quad n = 0,1,2,..$

I.Aref'eva, I.V. 2304.04695; 2305.19827

Equations of mathematical physics and negative dimensions

Areas and volumes

- **Cube** C^d. **Volume** $\mathcal{V}(C^d) = L^d$
- **Sphere** S^d. **Area** $\mathcal{A}(S^d) = \Omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2})}$
- **Ellipsoid** E^d: $\sum_1^{d+1} \frac{x_i^2}{a_i^2} \leq 1$. **Volume of** E^d:

$\mathcal{V}(E^d) = \frac{\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2}+1)} a_1 \cdot \ldots \cdot a_{d+1}$. **Take** $a_i = e^{c^i}$, **then**

$$\mathcal{V}(E^d) = \frac{\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2}+1)} e^{\sum_{i=1}^{d+1} c^i} = \frac{\pi^{(d+1)/2}}{\Gamma(\frac{d+1}{2}+1)} e^{c \frac{1-c^{d+1}}{1-c}}$$

Euler Gamma function

$$(1)$$

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx, \quad \Re s > 1 \qquad (2)$$

$$(3)$$

$$\Gamma(n) = (n-1)! \qquad (4)$$

$\Gamma(s)$ is a meromorphic function in $\mathbb{C}$ with simple poles at $s = 0, -1, -2, ...$

$\Gamma(s) \neq 0$ if $s \in \mathbb{C}$

Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1}, \quad \Re s > 1 \tag{5}$$

$\zeta(s)$ is a meromorphic function in $\mathbb{C}$ with simple a pole at $s = 1$

- **Zeros of $\zeta(s)$**
  - **"trivial zeros":** $s = -2, -4, ...$
  - **"non-trivial zeros"in** $0 < \Re s < 1$
  - **Riemann hypothesis: All non-trival zeros lie on line** $\Re s = \frac{1}{2}$ **(Hardy theorem)**

Functional Relations

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

$$\Gamma(s)\,\zeta(s) = \frac{\pi^s\,\zeta(1-s)}{2^{1-s}\sin(\frac{\pi(1-s)}{2})},$$

Laplace equation

- **Fundamental solution of the Laplace equation**

$$E_n(x) = \frac{r^{2-n}}{(2-n)\Omega_{n-1}}, \qquad \Delta E_n = \delta,$$

- **Fractional calculus**

$$\frac{x_+^{\lambda}}{\Gamma(\lambda+1)} * \frac{x_+^{\mu}}{\Gamma(\mu+1)} = \frac{x_+^{\lambda+\mu}}{\Gamma(\lambda+\mu+1)}$$

Why negative dimension?

- **Schwarzschild BH vs Bose gas**

$$S_{BH} = \frac{1}{16\pi T^2} \sim \beta^2$$

$$S_{BG} \sim \beta^{-d/2}$$

$$-d/2 = 2 \Rightarrow d = -4$$ - **negative dim.**

Cosmological Constant and Maximum of Entropy for de Sitter Space

- There are at least two cosmological constants calling for explanation
  - The first one describes the quasi-de Sitter inflation in the early universe
  - The second describes the current acceleration of the universe associated with dark energy
- The inflationary cosmological constant is computed

Inflationary cosmological constant

$$\Lambda = 3\pi \, \exp\{-\psi(3/2)\}$$

ψ **is the digamma function,**

$\Lambda \approx 9.087$ **in Planck units**

I.V., 2308.11377

Quantum gravity and cosmology

- **Wheeler DeWitt, Hartle-Hawking, Vilenkin, Linde,... At this conference: D. Minic**

$$< h_2, \phi_2, \Sigma_2 | h_1, \phi_1, \Sigma_1 >= \int \exp\left[iS(g,\phi) \right] \mathcal{D}g\mathcal{D}\phi$$

- **Tunneling, no-boundary**
- **Inflation**
- **De Sitter space**

De Sitter entropy

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \left(1 - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\omega^2_{D-2},$$

- $T = \frac{1}{2\pi\ell}$
- $S_{dS} = \frac{\beta^{D-2}}{4G_D(2\pi)^{D-2}}\Omega_{D-2}, \qquad F_{dS} = \frac{\beta^{D-3}}{4G_D(2\pi)^{D-2}(D-3)}\Omega_{D-2}$

Principle of maximal entropy

- **The principle of maximum entropy states that the probability distribution which best represents the current state of knowledge about a system is the one with largest entropy.**
- **We will use the following modification of this principle which takes into account the knowledge that the observed spacetime dimension is $D_0 = 4$.**
- **This modified principle can be called the conditional principle of maximum entropy(CPME)**

Conditional principle of maximum entropy (CPME)

- **CPME can be formulated as follows:**

  The dS radius ℓ_0 satisfies the condition that the maximum entropy $S(\ell_0, D)$ over the changing number of space-time dimensions is realized at $D = 4$.

  This maximal value coincides with the entropy value $S(\ell_0, D_0)$ for the observed spacetime dimension $D_0 = 4$,

  i.e. ℓ_0 such that

$$\ell_0 : \quad \max_D S(\ell_0, D) = S(\ell_0, D_0).$$

Computation of Λ 1/2

The entropy of de Sitter space is given by

$$S(\ell, D) = \frac{1}{4}\,\ell^{D-2}\Omega_{D-2} = \frac{1}{2}\,\ell^{D-2}\frac{\pi^{(D-1)/2}}{\Gamma((D-1)/2)}.$$

We consider D as a real variable. The derivative of entropy with respect to D is

$$\frac{\partial S(\ell, D)}{\partial D} = \frac{\pi^{\frac{D-1}{2}}\ell^{D-2}}{4\Gamma\left(\frac{D-1}{2}\right)}\left(\log(\pi\ell^2) - \psi\left(\frac{D-1}{2}\right)\right),$$

where ψ is the digamma function,

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z).$$

Computation of Λ 2/2

For extremum of the entropy with respect to D we have

$$\frac{\partial S(\ell, D)}{\partial D} = 0 \qquad \Rightarrow \qquad \log(\pi \ell^2) - \psi\left(\frac{D-1}{2}\right) = 0$$

Solving this equation for $D = 4$ with respect to ℓ_0, i.e.

$$\log(\pi \ell^2) - \psi\left(\frac{3}{2}\right) = 0$$

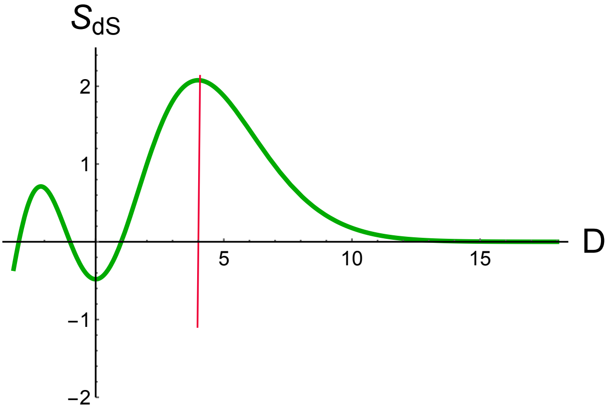
we find

$$\ell_0^2 = \frac{1}{\pi} \exp\{\psi(3/2)\} \approx 0.3301$$

i.e.

$$\Lambda = \frac{3}{\ell_0^2} = 3\pi \exp\{-\psi(3/2)\} \approx 9.087$$

S_{dS} as function of D for $\ell_0 = 0.575$



- The vertical red line shows the location of maximum entropy for $D = 4$
- Number of dimension plays the role of time

Spacetime dimension and the first law of thermodynamics

- **We interpret the spacetime dimension D as a thermodynamic quantity by analogy with the number of particles N**

- **We add the corresponding term to the first law of thermodynamics, which now reads**

$$dF = -S\,dT + \nu d\,D.$$

- **Here ν is a thermodynamics potential analogues to the chemical potential.**

Planck era

- We discussed the creation of the Universe from "nothing"after the Planck era.

- On the Planck scale, the usual consideration based on quantum gravity is not applicable. Since there are fluctuations in geometry, topology, and even number fields
- Fluctuations of number fields during Planck era

I.V.' 87;

Arefeva, Dragovich, Frampton, I.V.' 91

Dimension change, cobordism, entanglement

- **Quantum mechanics**
  - **1 particle in $\mathbb{R}^n$**

$$< y,t|x,0 >= \int e^{iS(q(t))}\mathcal{D}q(t), \quad q(0)=x, q(t)=y$$

$$x=(x_1,...x_n)\in\mathbb{R}^n, \quad y=(y_1,...y_n)\in\mathbb{R}^n$$

  - **Proposal with dimension change :** $\mathbb{R}^k\to\mathbb{R}^n, \quad k<n$
    Reduction (entanglement)

$$< y_1,...y_n,t|x_1,...x_k,0 >= \int < y_1,...y_n,t|x_1,...x_k,x_{k+1},...x_n,0 > dx_{k+1}...dx_n$$

Dimension change, cobordism, entanglement

- **Quantum gravity**

$$< h_2, \phi_2, \Sigma_2^{(n)} | h_1, \phi_1, \Sigma_1^{(n)} >= \int e^{iS(g,\phi)} \mathcal{D}g\mathcal{D}\phi$$

$$\partial M = \Sigma_1^{(n)} \cup \Sigma_2^{(n)} \quad \textbf{cobordism}$$

  - **Proposal.**
    Dimension change:

$$\Sigma_1^{(k)} \to \Sigma_2^{(n)}.$$

$$h_{\alpha\beta}(x^\gamma) = h_{\alpha\beta}(x^a, x^j)$$

$$< h_2, \phi_2, \Sigma_2 | h_1', \phi_1', \Sigma_1' >= \int < h_2, \phi_2, \Sigma_2 | h_1, \phi_1, \Sigma_1 > \mathcal{D}\sigma$$

General case

$$U_t \ : \ \mathcal{H}_1 \to \mathcal{H}_1 \qquad W : \mathcal{H}_1 \to \mathcal{H}_2$$

$$V_t = W\,U_t \ : \ \mathcal{H}_1 \to \mathcal{H}_2$$

$$L^2(\mathbb{R}^k) \to L^2(\mathbb{R}^n)$$

Conclusion

- **Equivalence of thermodynamics of D-dimensional Schwarzschild BH and Bose gas in negative dimension is established.**
  BH/BG-correspondence

- **The inflationary cosmological constant**

$$\Lambda = 3\pi \exp\{-\psi(3/2)\} \approx 9.087$$