The role of polyhedral products in geometric and topological combinatorics

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Related lectures

The role of polyhedral products in geometric and topological combinatorics.

- Combinatorics seminar
  Alfréd Rényi Institute of Mathematics
  Budapest, May 18, 2023
- International Polyhedral Products Seminar
  Princeton University Mathematics Department
  March 2, 2023

Recent developments

- Marinko Timotijević, Filip D. Jevtić, R. Ž, Polytopality of simple games, arXiv, September 2023.

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- Marinko Timotijević, Filip D. Jevtić, R. Ž, Polytopality of simple games, arXiv, September 2023.
- *Simple game* $\mathcal{G} = (P, \Gamma)$
  $\Gamma \subseteq 2^P$ the set of wining coalitions
  $K := 2^P \setminus \Gamma$ is the simplicial complex of losing coalitions
- Bier sphere $Bier(\mathcal{G}) = Bier(K) := K *_\Delta K^\circ$
- *Canonical fan* $Fan(\Gamma) = Fan(K)$.

Theorem 1

Theorem 1. Let $K \subsetneq 2^{[n]}$ be a proper simplicial complex such that $Vert(K) = [n]$. Then $\Gamma = 2^{[n]} \setminus K$ is a *roughly weighted simple game* with all weights strictly positive if and only if the canonical fan $Fan(\Gamma)$ of Γ is *pseudo-polytopal* in the sense that it refines the normal fan of a convex polytope.

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A simple game (P, Γ), where $K = 2^P \setminus \Gamma$ is the collection of losing coalitions, is *roughly weighted* if there exist strictly positive real numbers $w = (w_1, \dots, w_n)$ and a positive real number q (called the quota) such that for each $X \in 2^P$

$$w(X) = \sum_{i \in X} w_i < q \quad \Rightarrow \quad X \in K \tag{1}$$

$$w(X) = \sum_{i \in X} w_i > q \quad \Rightarrow \quad X \in \Gamma \tag{2}$$

Theorem 2

Theorem 2. All Bier spheres with up to ten vertices are polytopal. There are 88 non-threshold complexes on 5 vertices, and 48 corresponding non-isomorphic Bier spheres all polytopal. An example of such a sphere is *Bier*(**Möb**) where **Möb** is the minimal triangulation of the Möbius band.

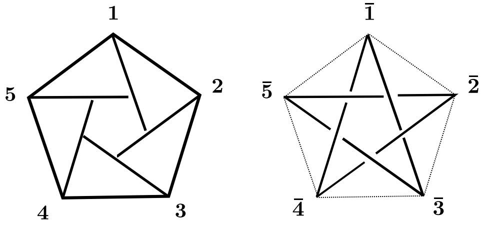

Figure: Triangulated Möbius band as a dual of a 5-cycle.

Known results

It is known [JTZ19] [JZ23] that the Bier spheres of threshold complexes (weighted majority games) are polytopal.

- All simplicial 3-spheres with up to 7 vertices are polytopal.
- The Grünbaum-Sreedharan sphere and the Barnette sphere are the only two 3-spheres with 8 vertices which are non-polytopal.
- The classification of 3-spheres with 9 vertices into polytopal and non-polytopal spheres, started by Altshuler and Steinberg, completed by Altshuler, Bokowski, and Steinberg, see [Lutz08] for the references.
- The classification of 3-spheres with 10 vertices (open)!?

$\mathcal{Z}_K(X, A) = (X, A)^K$

Let (X, A) be a pair of spaces and let K be an abstract simplicial complex, $K \subseteq 2^{[n]}$.

The associated *Polyhedral Product* (generalized moment-angle complex, K-power) is the space,

$$(X, A)^K = \mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma = \bigcup_{\sigma \in K} \Big(\prod_{i \in \sigma} X \times \prod_{j \notin \sigma} A\Big) \subseteq X^n.$$

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- $\mathcal{Z}_K(D^2,S^1)$ moment-angle complex (toric topology);
- $\mathcal{Z}_K(D^1,S^0)$ small cover;
- $(\mathbb{C}P^\infty)^K$ Davis–Januszkiewicz space, etc.

Victor M. Buchstaber, Taras E. Panov. Toric Topology, A.M.S. 2015.

$\mathcal{Z}_K(X, A)$

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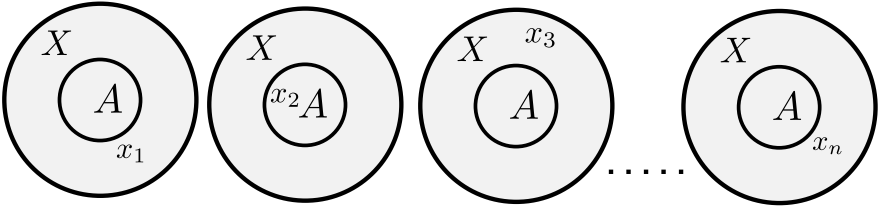
For $x = (x_i) \in X^n$ let $M_A(x) := \{i \in [n] \mid x_i \notin A\}$.

Then

$$\mathcal{Z}_K(X, A) := \{x \in X^n \mid M_A(x) \in K\}.$$

$\mathcal{Z}_K(X, A)$



$$M_A(x) := \{i \in [n] \mid x_i \notin A\} \quad S_A := [n] \backslash M_A = \{j \in [n] \mid x_i \in A\}\,.$$

Alan D. Taylor and William S. Zwicker. Simple Games: Desirability Relations, Trading, Pseudoweightings. Princeton University Press, 1999.

Edmonds-Fulkerson bottleneck thm.

Bottleneck Extrema

JACK EDMONDS AND D. R. FULKERSON

National Bureau of Standards, Washington, D.C. 20234, and
The RAND Corporation, 1700 Main Street, Santa Monica, California 90406

Communicated by W. T. Tutte

Received March 11, 1968

ABSTRACT

Let E be a finite set. Call a family of mutually noncomparable subsets of E a clutter on E. It is shown that for any clutter $\mathscr{R}$ on E, there exists a unique clutter $\mathscr{S}$ on E such that, for any function f from E to real numbers,

$$\min_{R \in \mathscr{R}} \max_{x \in R} f(x) = \max_{S \in \mathscr{S}} \min_{x \in S} f(x).$$

Specifically, $\mathscr{S}$ consists of the minimal subsets of E that have non-empty intersection with every member of $\mathscr{R}$. The pair $(\mathscr{R}, \mathscr{S})$ is called a blocking system on E. An algorithm is described and several examples of blockings systems are discussed.

Bier sphere $B(K, K^\circ)$

$$K * L = \{A \uplus C \mid A \in K, C \in L\}\,.$$

$$K *_\Delta L = \{A \uplus C \mid A \in K, C \in L \text{ and } A \cap C = \emptyset\}\,.$$

$$K^\circ = \{A \subset [m] \mid A^c \notin K\} \qquad \text{is the Alexander dual of } K\,.$$

$$Bier(K) = B(K, K^\circ) := K *_\Delta K^\circ$$

is the associated *Bier sphere*.

Bottleneck theorem and discrete Morse theory

$$\min_{A\in\mathcal{A}}\max_{x\in A} f(x) = \max_{B\in\mathcal{B}}\min_{x\in B} f(x) = f(c) \qquad (3)$$

Let $K := 2^{[n]} \setminus \mathcal{A}$ and $L = K^\circ := 2^{[n]} \setminus \mathcal{B}$ and let $Bier(K) = K *_\Delta K^\circ \cong S^{n-2}$ be the associated Bier sphere. Then $f : [n] \to \mathbb{R}$ (assumed to be 1–1) induces a perfect (discrete) Morse function on $Bier(K)$ with the critical cell in dimension $(n-2)$ of the form $(X, c, Y) \in Bier(K) = K *_\Delta K^\circ$.

D. Jojić, G. Panina, S. Vrećica, R. Živaljević. Generalized chessboard complexes and discrete Morse theory. Chebyshevskii Sbornik, 2020, Volume 21, Issue 2, 207–227.

Bier sphere $B(K, K^\circ)$

If $Vert(K) = [n] = \{1, 2, \ldots n\}$, $Vert(K^\circ) = [\bar{n}] = \{\bar{1}, \bar{2}, \ldots \bar{n}\}$
then $Vert(B(K, K^\circ)) = [n] \cup [\bar{n}]$ and
$(A, B, C) \in B(K, K^\circ)$ is equivalent to

- $[n] = A \uplus B \uplus C$ (disjoint union);
- $A \in K$ and $\bar{C} := \{\bar{k}\}_{k \in C} \in K^\circ$;
- $\emptyset \neq B \neq [n]$.

Alexander duality revisited

Alexander duality for generalized moment-angle complexes.

Proposition: (V. Welker, V. Grujić)

$$\mathcal{Z}_K(X, A) \uplus \mathcal{Z}_{K^\circ}(X, A^c) = X^m \,.$$

Proof: For each $x \in X^m$ either $M_A(x) \in K$ or $M_{A^c}(x) \in K^\circ$, but not both! Indeed, $M_A(x) \cap M_{A^c}(x) = \emptyset$ and $M_A(x) \cup M_{A^c}(x) = [m]$.

$\mathcal{Z}_K(I, I_{\leqslant \frac{1}{2}}) \cap \mathcal{Z}_{K^\circ}(I, I_{\geqslant \frac{1}{2}})$

Let $I = [0,1]$, $I_{\leqslant \frac{1}{2}} := [0, \frac{1}{2}]$, $I_{\geqslant \frac{1}{2}} := [\frac{1}{2}, 1]$, $I_{< \frac{1}{2}} := [0, \frac{1}{2})$, etc.

Similarly, for $J = [-1, 1]$
$J_{\leqslant 0} := [-1, 0]$, $J_{\geqslant 0} := [0, 1]$, $I_{<0} := [-1, 0)$, etc.

Question:

$$\mathcal{Z}_K(I, I_{\leqslant \frac{1}{2}}) \cap \mathcal{Z}_{K^\circ}(I, I_{\geqslant \frac{1}{2}}) =: Z(K, K^\circ) =? \,.$$

Proposition:

$$Z(K, K^\circ) = \bigcup_{(A,B,C) \in B(K,K^\circ)^+} (I_{\geqslant \frac{1}{2}})^A \times \ \{\tfrac{1}{2}\}^B \ \times \ (I_{\leqslant \frac{1}{2}})^C \ \ .$$

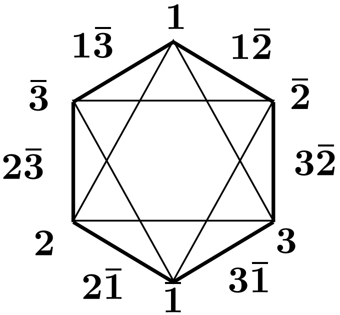
where $B(K, K^\circ)^+ := B(K, K^\circ) \cup \{(\emptyset, [n], \emptyset)\}$.

$\partial Z(K, K^\circ) \cong B(K, K^\circ)$

Proposition:

- $B(K, K^\circ)$ is a triangulation of S^{n-2};
- $Z(K, K^\circ) \cong D^{n-1}$;
- $Z(K, K^\circ)$ is a cubification (cubical complex) on $\cong D^{n-1}$;
- $\partial Z(K, K^\circ)$ is a quadrangulation (cubification) of S^{n-2}.
- $\partial Z(K, K^\circ)$ is the *canonical cubification* of $B(K, K^\circ)$.

$\partial Z(K, K^\circ) \cong B(K, K^\circ)$



$$K = Vert(K) = \{1, 2, 3\} \quad K^\circ = Vert(K) = \{\bar{1}, \bar{2}, \bar{3}\}$$

$$Bier(K, K^\circ) = \{1\bar{2}, 3\bar{2}, 3\bar{1}, 2\bar{1}, 2\bar{3}, 1\bar{3}\}$$

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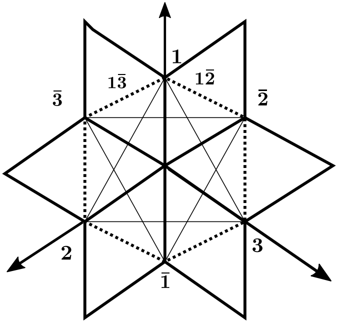

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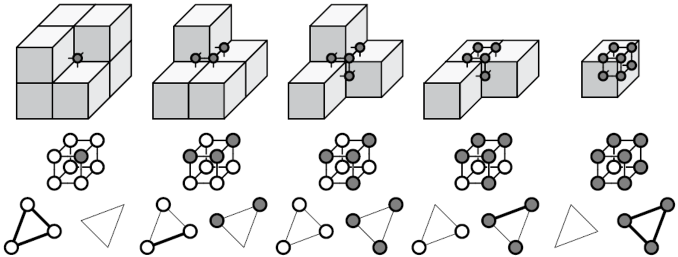
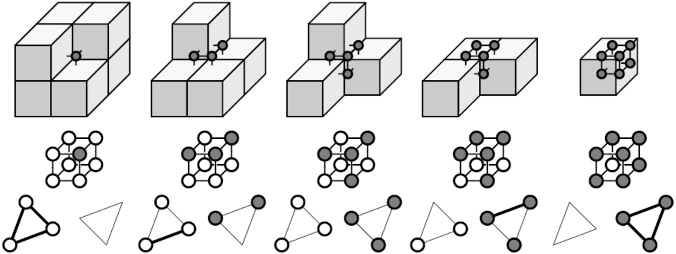
$$Z(K,K^\circ) = \bigcup_{(A,B,C)\in B(K,K^\circ)^+} (I_{\geqslant \frac{1}{2}})^A \times \{\tfrac{1}{2}\}^B \times (I_{\leqslant \frac{1}{2}})^C \ .$$

Examples of $\mathcal{Z}_K(I, I_{\leqslant\frac{1}{2}}) \cap \mathcal{Z}_{K^\circ}(I, I_{\geqslant\frac{1}{2}})$



Dave Bayer. Monomial Ideals and Duality. Barnard College and M.S.R.I. bayer@math.columbia.edu, February 8, 1996

Examples of $\mathcal{Z}_K(I, I_{\leqslant \frac{1}{2}}) \cap \mathcal{Z}_{K^\circ}(I, I_{\geqslant \frac{1}{2}})$



Dave Bayer. Monomial Ideals and Duality. Barnard College and M.S.R.I. bayer@math.columbia.edu, February 8, 1996

Ezra Miller, Bernd Sturmfels. Combinatorial Commutative Algebra. Springer, 2004.

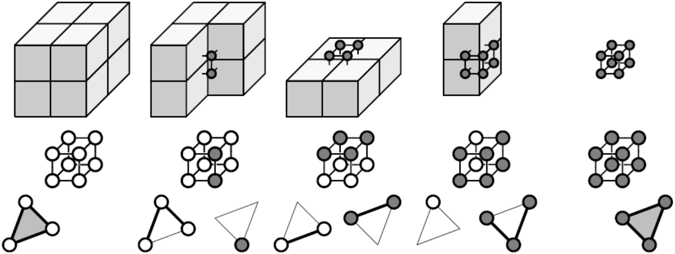


Figure 9: Corners and noncorners in 3 dimensions.

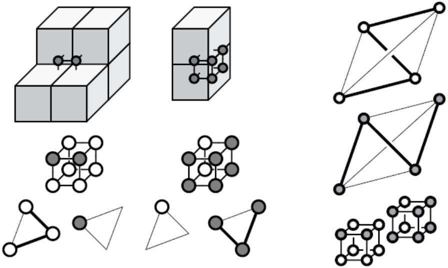


Figure 10: A corner with no homology, in 4 dimensions.

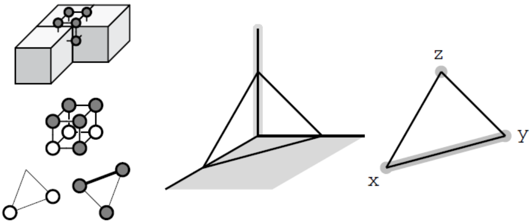


Figure 11: An observer's view of a corner.

Simplicial Steinitz problem and Bier spheres

The problem of deciding if a given triangulation of a sphere is realizable as the boundary sphere of a simplicial, convex polytope is known as the "Simplicial Steinitz problem"

G. Ewald: *Combinatorial Convexity and Algebraic Geometry*, volume 168 of Graduate Texts in Mathematics. Springer-Verlag, 1996.

Vast majority of *Bier spheres* $B(K, K^{\circ})$ are "non-polytopal", in the sense that they are not combinatorially isomorphic to the boundary of a convex polytope.

[1] A. Björner, A. Paffenholz, J. Sjöstrand, and G. M. Ziegler: *Bier spheres and posets.* Discrete & Computational Geometry, **34** (2004), No. 1, 71–86.

[2] S.Lj. Čukić, E. Delucchi. Simplicial shellable spheres via combinatorial blowups, *Proc. Amer. Math. Soc.* 135 (2007), no. 8, 2403–2414.

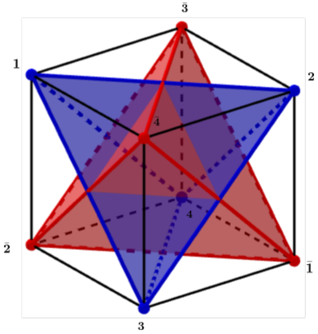
[3] E. Delucchi, L. Hoessly. Fundamental polytopes of metric trees via hyperplane arrangements, arXiv:1612.05534 [math.CO].

[4] F. D. Jevtić, M. Timotijević, and R. T. Živaljević: *Polytopal Bier spheres and Kantorovich–Rubinstein polytopes of weighted cycles.* Discrete & Computational Geometry, **65** (2019), No. 4, 1275–1286.

[5] D. Jojić, I. Nekrasov, G. Panina, R. Živaljević, Alexander r-tuples and Bier complexes, *Publ. Inst. Math. (Beograd) (N.S.)* 104(118) (2018), 1–22.
[6] D. Jojić, G. Panina, and R. Živaljević: *A Tverberg type theorem for collectively unavoidable complexes.* Israel J. Math. (2021).
[7] M. de Longueville. Bier spheres and barycentric subdivision, *J. Comb. Theory Ser. A* 105 (2004), 355–357.
[8] J. Matoušek: *Using the Borsuk–Ulam Theorem*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.
[9] M. Timotijević. Note on combinatorial structure of self-dual simplicial complexes, *Mat. Vesnik*, 71:104–122, 2019.

$B(K, K^\circ)$

[10] F. D. Jevtić, R. T. Živaljević. Bier spheres of extremal volume and generalized permutohedra. *Applicable Analysis and Discrete Mathematics*, 2022.



Braid arrangement

The braid arrangement is the arrangement of hyperplanes $\mathit{Braid}_n = \{H_{i,j}\}_{1 \leq i < j \leq n}$ in H_0 where $H_0 := \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^n/(1, \ldots, 1)\mathbb{R}$ and $H_{i,j} := \{x \mid x_i - x_j = 0\}$.

The hyperplanes $H_{i,j}$ subdivide the space H_0 into the polyhedral cones

$$C_\pi := \{x \in H_0 \mid x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}\}$$

labeled by permutations $\pi \in S_n$.

The cones C_π, together with their faces, form a complete simplicial fan in H_0, called the *braid arrangement fan*.

The preposet $\leftrightarrow$ braid cone dictionary

3.4 THE DICTIONARY

Let us say that a *braid cone* is a polyhedral cone in the space $\mathbb{R}^n/(1,\ldots,1)\mathbb{R} \simeq \mathbb{R}^{n-1}$ given by a conjunction of inequalities of the form $x_i - x_j \geq 0$. In other words, braid cones are polyhedral cones formed by unions of Weyl chambers or their lower dimensional faces.

There is an obvious bijection between preposets and braid cones. For a preposet Q on the set $[n]$, let σ_Q be the braid cone in the space $\mathbb{R}^n/(1,\ldots,1)\mathbb{R}$ defined by the conjunction of the inequalities $x_i \leq x_j$ for all $i \preceq_Q j$. Conversely, one can always reconstruct the preposet Q from the cone σ_Q by saying that $i \preceq_Q j$ whenever $x_i \leq x_j$ for all points in σ_Q.

3.3 PREPOSETS, EQUIVALENCE RELATIONS, AND POSETS

Recall that a *binary relation* R on a set X is a subset of $R \subseteq X \times X$. A *preposet* is a reflexive and transitive binary relation R, that is $(x,x) \in R$ for all $x \in X$, and whenever $(x,y),(y,z) \in R$ one has $(x,z) \in R$. In this case we will often use the notation $x \preceq_R y$ instead of $(x,y) \in R$. Let us also write $x \prec_R y$ whenever $x \preceq_R y$ and $x \neq y$.

The preposet $\leftrightarrow$ braid cone dictionary

A. Postnikov, V. Reiner, and L. Williams. *Faces of Generalized Permutohedra,* Documenta Mathematica, Vol. 13 (2008), 207–273.

PROPOSITION 3.5. *Let the cones σ, σ' correspond to the preposets Q, Q' under the above bijection. Then*

(1) The preposet $\overline{Q \cup Q'}$ corresponds to the cone $\sigma \cap \sigma'$.

(2) The preposet Q is a contraction of Q' if and only if the cone σ is a face σ'.

(3) The preposets Q, Q' intersect properly if and only if the cones σ, σ' do.

(4) Q is a poset if and only if σ is a full-dimensional cone, i.e., $\dim \sigma = n-1$.

The preposet $\leftrightarrow$ braid cone dictionary

(5) *The equivalence relation* $\equiv_Q$ *corresponds to the linear span* $\mathrm{Span}(\sigma)$ *of* σ.

(6) *The poset* $Q/\equiv_Q$ *corresponds to a full-dimensional cone inside* $\mathrm{Span}(\sigma_Q)$.

(7) *The preposet* Q *is connected if and only if the cone* σ *is pointed.*

(8) *If* Q *is a poset, then the minimal set of inequalities describing the cone* σ *is* $\{x_i \leq x_j \mid i \lessdot_Q j\}$. *(These inequalities associated with covering relations in* Q *are exactly the facet inequalities for* σ.*)*

(9) Q *is a tree-poset if and only if* σ *is a full-dimensional simplicial cone.*

(10) *For* $w \in \mathfrak{S}_n$, *the cone* σ *contains the Weyl chamber* C_w *if and only if* Q *is a poset and* w *is its linear extension, that is* $w(1) \prec_Q w(2) \prec_Q \cdots \prec_Q w(n)$.

The preposet $\leftrightarrow$ braid cone dictionary

According to Proposition 3.5, a full-dimensional braid cone σ associated with a poset Q can be described in three different ways (via all relations in Q, via covering relations in Q, and via linear extensions $\mathcal{L}(Q)$ of Q) as

$$\sigma = \{x_i \leq x_j \mid i \preceq_Q j\} = \{x_i \leq x_j \mid i \lessdot_Q j\} = \bigcup_{w \in \mathcal{L}(Q)} C_w.$$

COROLLARY 3.6. *A complete fan of braid cones (resp., pointed braid cones, simplicial braid cones) in* $\mathbb{R}^n/(1,\ldots,1)\mathbb{R}$ *corresponds to a complete fan of posets (resp., connected posets, tree-posets) on* $[n]$.

For a generalized permutohedron P, define the *vertex poset* Q_v at a vertex $v \in \mathrm{Vertices}(P)$ as the poset on $[n]$ associated with the normal cone $\mathcal{N}_v(P)/(1,\ldots,1)\mathbb{R}$ at the vertex v, as above.

COROLLARY 3.7. *For a generalized permutohedron (resp., simple generalized permutohedron)* P, *the collection of vertex posets* $\{Q_v \mid v \in \mathrm{Vertices}(P)\}$ *is a complete fan of posets (resp., tree-posets).*

The preposet $\leftrightarrow$ braid cone dictionary

Thus normal fans of generalized permutohedra correspond to certain complete fans of posets, which we call *polytopal*. In [M–W'06], the authors call such fans *submodular rank tests*, since they are in bijection with the faces of the cone of submodular functions. That cone is precisely the deformation cone we discuss in the Appendix.

COROLLARY 3.9. *Let P be a generalized permutohedron in $\mathbb{R}^n$, and $v \in \text{Vertices}(P)$ be a vertex. For $w \in \mathfrak{S}_n$, one has $\Psi_P(w) = v$ whenever the normal cone $\mathcal{N}_v(P)$ contains the Weyl chamber C_w. The preimage $\Psi_P^{-1}(v) \subseteq \mathfrak{S}_n$ of a vertex $v \in \text{Vertices}(P)$ is the set $\mathcal{L}(Q_v)$ of all linear extensions of the vertex poset Q_v.*

Canonical fan $Fan(K)$

Let $\tau = (A_1, B, A_2) \in B(K, K^\circ)$. Following [4] and [10], the associated *braid cone* is

$$Cone(\tau) = \{x \in H_0 \mid \begin{array}{l} x_i \leq x_j \text{ for each } (i,j) \in A_1 \times B \cup B \times A_2, \\ x_i = x_j \text{ for each } (i,j) \in B \times B \} . \end{array}$$

Theorem: Let $K \subsetneq 2^{[n]}$ be a simplicial complex. Then the collection of convex cones

$$Fan(K) = \{Cone(\preccurlyeq_\tau)\}_{\tau \in B(K,K^\circ)} \tag{4}$$

is a complete simplicial fan in $H_0 = \{x \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$, referred to as the *canonical fan* associated to K.

Canonical fan $Fan(K)$

Moreover, the face poset $FaceFan(K)$ is isomorphic to the (extended) face poset

$$Face(B(K, K^\circ)^+) := Face(B(K, K^\circ)) \cup \{\emptyset\}$$

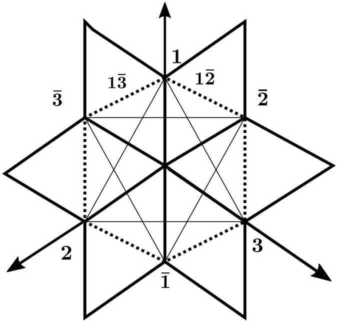
of the Bier sphere $B(K, K^\circ)$.

The construction of the canonical fan is faithful in the sense that if $Fan(K_1) = Fan(K_2)$ then $K_1 = K_2$.

Remark:

$$Fan(K) \quad \overset{flattening}{\longleftrightarrow} \quad \mathcal{Z}_K(\mathbb{R}, \mathbb{R}_{\leqslant 0}) \cap \mathcal{Z}_{K^\circ}(\mathbb{R}, \mathbb{R}_{\geqslant 0})$$

$\partial Z(K, K^\circ) \cong B(K, K^\circ)$



$$K = Vert(K) = \{0, 1, 2\} \quad K^\circ = Vert(K) = \{\bar{0}, \bar{1}, \bar{2}\}$$

$$B(K, K^\circ) = \{1\bar{2}, 3\bar{2}, 3\bar{1}, 2\bar{1}, 2\bar{3}, 1\bar{3}\}$$

$$Z(K, K^\circ) = \bigcup_{(A,B,C)\in B(K,K^\circ)^+} (I_{\geqslant \frac{1}{2}})^A \times \{\tfrac{1}{2}\}^B \times (I_{\leqslant \frac{1}{2}})^C .$$

Corollary: Each Bier sphere $Bier(K)$, defined as a canonical triangulation of a $(n-2)$ sphere S^{n-2} associated to an abstract simplicial complex $K \subsetneq 2^{[n]}$, admits a starshaped embedding in $\mathbb{R}^{n-1}$.

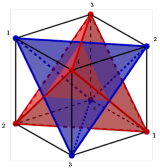


Figure: The 3-dimensional cube as the Van Kampen-Flores polytope Ω_4.

Glossary

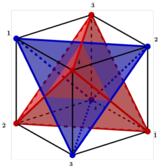
$B(K, K^\circ) = K *_\Delta K^\circ$, the Bier sphere of K, is a combinatorial object (deleted join of two simplicial complexes).

Fan(K), the *canonical* of K, is a complete, simplicial fan in $H_0 \cong \mathbb{R}^{n-1}$, associated to a simplicial complex $K \subsetneq 2^{[n]}$.

$\mathcal{R}_{\pm\delta}(B(K, K^\circ))$, *canonical starshaped realization* of $B(K, K^\circ)$.

Star(K) the body whose boundary is the sphere $\mathcal{R}_{\pm\delta}(B(K, K^\circ))$.

Ω_n is a universal, $(n-1)$-dimensional convex polytope (the Van Kampen-Flores polytope) which is equal, as a convex body, to *Star*(K) for each Bier sphere of maximal volume.



Bier spheres of maximal volume

Proposition: Assume that $K \subsetneq 2^{[n]}$ is a simplicial complex and let $Star(K) \subset H_0$ be the associated starshaped body. Let $B \notin K$ be a minimal non-face of K in the sense that $(\forall i \in B)\, B \setminus \{i\} \in K$, and let $K' = K \cup \{B\}$. Let $C = [n] \setminus B$ the complement of B. Then

$$Vol\,(Star(K')) - Vol\,(Start(K)) = V(K', K) = (|C| - |B|)Vol_0\,.$$

The following relations are an immediate consequence

$$V(K', K) > 0, \text{ if } |B| < \frac{n}{2}$$
$$V(K', K) = 0, \text{ if } |B| = \frac{n}{2}$$
$$V(K', K) < 0, \text{ if } |B| > \frac{n}{2}$$

Bier spheres of maximal volume

Theorem: If $n = 2m + 1$ is odd the unique Bier sphere of maximal volume is $B(K, K^\circ)$ where

$$K = \binom{[n]}{\leq m} = \{S \subset [n] \mid |S| \leq m\}\,. \tag{5}$$

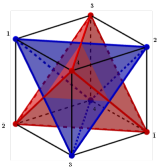
If $n = 2m$ is even a Bier sphere $B(K, K^\circ)$ is of maximal volume if and only if

$$\binom{[n]}{\leq m-1} \subseteq K \subseteq \binom{[n]}{\leq m}\,. \tag{6}$$

A Bier sphere $B(K, K^\circ)$ is of minimal volume if and only if either $K = \{\emptyset\}$ or K is the boundary of the simplex $\Delta_{[n]}$, $K = \partial\Delta_{[n]} = 2^{[n]} \setminus \{[n]\}$.

Universal Bier polytope Ω_n

Corollary: For all Bier spheres $Bier(K)$ of maximal volume, the convex body $\Omega_n = Star(K)$ is unique and independent of K. The body Ω_n is centrally symmetric. More explicitly $\Omega_n = \mathrm{Conv}(\Delta_\delta \cup \nabla_\delta)$ where $\Delta_\delta \subset H_0$ is the simplex spanned by vertices $\delta_i := e_i - \frac{1}{n}(e_1 + \cdots + e_n)$ and $\nabla_\delta := -\Delta_\delta = \Delta_{\bar{\delta}}$ is the simplex spanned by $\bar{\delta}_i = -\delta_i$. The centrally symmetric $(n-1)$-dimensional convex body Ω_n may be (informally) referred to as the *Van Kampen-Flores-Bier polytope* in dimension $n-1$.



Balanced complexes and Van Kampen-Flores polytope Ω_n

Theorem: (D. Jojić, G. Panina, R.Ž; Israel J. Math. (2021)) Let $K \subset 2^{[n]}$ be a simplicial complex and let K° be its Alexander dual. Let $n = 2m$ and assume that K is **balanced** in the sense that

$$\binom{[n]}{\leq m-1} \subseteq K \subseteq \binom{[n]}{\leq m}. \tag{7}$$

Then for each continuous map $f : \Delta^{n-1} \to \mathbb{R}^{n-3}$ there exist disjoint faces $F_1 \in K$ and $F_2 \in K^\circ$ such that $f(F_1) \cap f(F_2) \neq \emptyset$.

The importance of balanced complexes was noted even earlier. In [Matoušek, 2008] they were used as a source of examples of non-polytopal triangulations of spheres while in [Björner et al. (2004)] they provided examples of *nearly neighborly Bier spheres*.

Connection with hypersimplices

Definition: A hypersimplex $\Delta_{n,r}$ with parameters n, r is defined as the convex hull of all n-dimensional vectors, vertices of the n-dimensional cube $[0,1]^n$, which belong to the hyperplane $x_1 + \cdots + x_n = r$.

Alternatively $\Delta_{n,r} = \mathrm{Newton}(\sigma_r)$ can be described as the Newton polytope of the elementary symmetric function σ_r of degree r in n variables.

$(\Omega_n)^\circ$ is a hypersimplex

Proposition: If $n = 2k$ is even then $\Omega^\circ_{2k} = \Delta \cap \nabla$ is affine isomorphic to the hypersimplex $\Delta_{2k,k}$. If $n = 2k+1$ then Ω°_n is affine isomorphic to the convex hull

$$\Omega^\circ_{2k+1} \cong \mathrm{Conv}\{\lambda \in [0,1]^{2k+1} \mid \begin{array}{l} (\forall i)\ \lambda_i \in \{0, \frac{1}{2}, 1\} \text{ and} \\ |Z(\lambda)| = |W(\lambda)| = k \end{array} \} \tag{8}$$

where $Z(\lambda) = \{j \mid \lambda_j = 0\}$ and $W(\lambda) = \{j \mid \lambda_j = 1\}$.

Steinitz problem for Bier spheres

Theorem: Let $\mathcal{F} = Fan(K)$ be the radial fan arising from the canonical starshaped realization of the associated Bier sphere $Bier(K)$. Then $\mathcal{F}$ is a normal fan of a convex polytope if and only if the simplicial complex K admits a K-submodular function. Moreover, there is a bijection between convex realizations of $Bier(K)$ with radial fan $\mathcal{F}$ and K-submodular functions f.

Corollary: Bier sphere $Bier(T_{\mu_L<\nu})$ of a threshold complex $T_{\mu_L<\nu}$ is isomorphic to the boundary sphere of a convex polytope which can be realized as a polar dual of a generalized permutohedron.

K-submodular functions, associated to $Bier(K, K^\circ)$

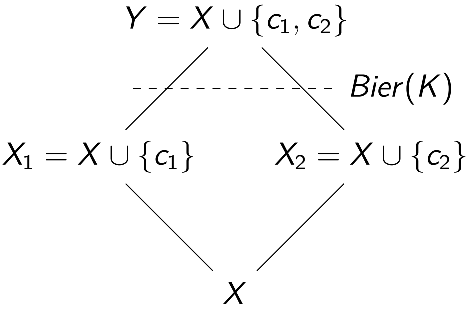
Definition: Let $K \subsetneq 2^{[n]}$ be a simplicial complex and $Bier(K)$ the associated Bier sphere. A *K-submodular function* (K-wall crossing function) is a function $f : Vert(Bier(K)) \to \mathbb{R}$ such that

$$f(c_1) + f(c_2) + \Sigma_{i\in X} f(i) > \Sigma_{j\notin Y} f(\bar{j}) \quad \text{for each } \Lambda\text{-configuration} \tag{9}$$

$$f(\bar{c}_1) + f(\bar{c}_2) + \Sigma_{j\notin X} f(\bar{j}) > \Sigma_{i\in X} f(i) \quad \text{for each } V\text{-configuration} \tag{10}$$

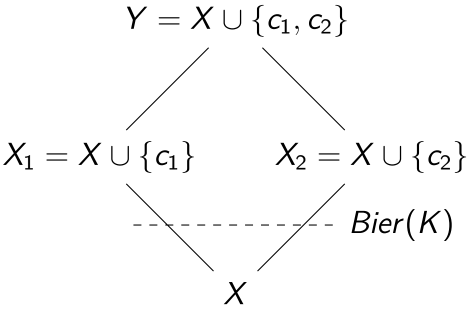
$$f(c_2) + f(\bar{c}_2) > 0 \quad \text{for each } X\text{-configuration.} \tag{11}$$

K-submodular functions



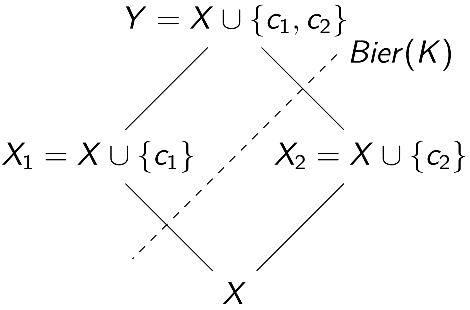
[Λ configuration]

K-submodular functions



[V configuration]

K-submodular functions



[X configuration]

K-submodular functions

Definition: Let $K \subsetneq 2^{[n]}$ be a simplicial complex and $Bier(K)$ the associated Bier sphere. A *K-submodular function* (K-wall crossing function) is a function $f : Vert(Bier(K)) \to \mathbb{R}$ such that

$$f(c_1) + f(c_2) + \Sigma_{i \in X} f(i) > \Sigma_{j \notin Y} f(\bar{j}) \quad \text{for each } \Lambda\text{-configuration} \tag{12}$$

$$f(\bar{c}_1) + f(\bar{c}_2) + \Sigma_{j \notin X} f(\bar{j}) > \Sigma_{i \in X} f(i) \quad \text{for each } V\text{-configuration} \tag{13}$$

$$f(c_2) + f(\bar{c}_2) > 0 \quad \text{for each } X\text{-configuration.} \tag{14}$$

Strong polytopality of Bier spheres

Theorem.([10]) Let $\mathcal{F} = Fan(K)$ be the radial fan arising from the canonical starshaped realization of the associated Bier sphere $Bier(K)$. (Recall that $\mathcal{F}$ is refined by the braid arrangement fan.) Then $\mathcal{F}$ is a normal fan of a convex polytope if and only if the simplicial complex K admits a K-submodular function. Moreover, there is a bijection between convex realizations of $Bier(K)$ with radial fan $\mathcal{F}$ and K-submodular functions f.

Proposition. Let $\mathcal{F}$ be an essential complete simplicial fan in $\mathbb{R}^n$ and $\mathbf{G}$ be the $N \times n$ matrix whose rows are the rays of $\mathcal{F}$. Then the following are equivalent for any vector $\mathbf{h} \in \mathbb{R}^N$.

(I) The fan $\mathcal{F}$ is the normal fan of the polytope $P_{\mathbf{h}} := \{x \in \mathbb{R}^n \mid \mathbf{G}x \leq \mathbf{h}\}$.

(II) For any two adjacent chambers $\mathbb{R}_{\geqslant 0}\mathbf{R}$ and $\mathbb{R}_{\geqslant 0}\mathbf{S}$ of $\mathcal{F}$ with $\mathbf{R} \setminus \{r\} = \mathbf{S} \setminus \{s\}$,

$$\alpha \mathbf{h_r} + \beta \mathbf{h_s} + \sum_{\mathbf{t} \in \mathbf{R} \cap \mathbf{S}} \gamma_{\mathbf{t}} \mathbf{h_t} > 0, \tag{15}$$

where

$$\alpha \mathbf{r} + \beta \mathbf{s} + \sum_{\mathbf{t} \in \mathbf{R} \cap \mathbf{S}} \gamma_{\mathbf{t}} \mathbf{t} = 0 \tag{16}$$

is the unique (up to scaling) linear dependence with $\alpha, \beta > 0$ between the rays of $\mathbf{R} \cup \mathbf{S}$.

Strong polytopality of threshold complexes

Recall that $T_{\mu_L<\nu} := \{I \subseteq [n] \mid \mu_L(I) < \nu\}$ is a threshold complex where $L = (l_1, l_2, \dots, l_n)$ is a (strictly positive) vector of weights such that $l_1 + \cdots + l_n = 1$. Assuming (w.l.o.g.) that $\mu_L(I) \neq \nu$ for each $I \subseteq [n]$, the Alexander dual of K is $K^\circ = T_{\mu_L \leq 1-\nu} = T_{\mu_L<1-\nu}$.

Corollary. ([4] and [10]) *Bier*$(T_{\mu_L<\nu})$ is isomorphic to the boundary sphere of a convex polytope which can be realized as a polar dual of a generalized permutohedron.

Proof (outline)

Construct a K-submodular function $f : [n] \cup [\bar{n}] \to \mathbb{R}$ where $[n] \cup [\bar{n}] = Vert(Bier(K))$. The function defined by

$$f(i) = (1-\nu)l_i \qquad f(\bar{j}) = \nu l_j \qquad (i, j = 1, \ldots, n) \qquad (17)$$

is indeed K-submodular for $K = T_{\mu_L < \nu}$. The inequalities (12) and (13), for the function f defined by (17), take the following form

$$\nu\mu_L(Y) > (1-\nu)\mu_L(Y^c) \qquad (1-\nu)\mu_L(X) < \nu\mu_L(X^c)\,. \quad (18)$$

However, in a threshold complex, both inequalities (18) hold without any restrictions on a simplex $X \in K$ and a non-simplex $Y \notin K$. (For example the second inequality in (18) is a consequence of $\mu_L(X) < \nu$ and $\mu(X^c) > 1 - \nu$.)

Converse is also true!

Theorem. (February, 2023) $Bier(K, K^{\circ})$ is strongly polytopal (i.e. there exists a K-submodular function) if and only if K is a threshold complex.

Proof by a direct construction of a weight distribution $L = (l_1, l_2, \ldots, l_n) \in \mathbb{R}^n_+$ from a K-submodular function $f : [n] \cup [\bar{n}] \to \mathbb{R}$.

Polyhedral products and generalized Van Kampen-Flores theorems

Theorem: (D. Jojić, G. Panina, R.Ž; Israel J. Math. (2021)) Let $K \subset 2^{[n]}$ be a simplicial complex and let K° be its Alexander dual. Let $n = 2m$ and assume that K is **balanced** in the sense that

$$\binom{[n]}{\leq m-1} \subseteq K \subseteq \binom{[n]}{\leq m}. \tag{19}$$

Then for each continuous map $f : \Delta^{n-1} \to \mathbb{R}^{n-3}$ there exist disjoint faces $F_1 \in K$ and $F_2 \in K^\circ$ such that $f(F_1) \cap f(F_2) \neq \emptyset$.

Collectively unavoidable complexes

Definition: An ordered r-tuple $\mathcal{K} = \langle K_1, \dots, K_r \rangle$ of subcomplexes of $2^{[m]}$ is *collectively r-unavoidable* if for each **ordered collection** $(A_1, \dots, A_r)$ of disjoint sets in $[m]$ there exists i such that $A_i \in K_i$.

Example: The pair $\langle K, K^{\circ} \rangle$ is collectively unavoidable.

A complex $K \subseteq 2^{[r]}$ is by definition *r-unavoidable* if the r-tuple $\langle K, K, \dots, K \rangle$ is collectively r-unavoidable.

Van Kampen-Flores type theorem for collectively unavoidable complexes

Theorem A. $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \dots, K_r \rangle$ is a *collectively r-unavoidable* family of subcomplexes of the N-dimensional simplex $\Delta_N = 2^{[N+1]}$, where $r = p^\nu$ is a power of a prime.

Assume that there exists $k \geq 1$ such that for each i

$$\Delta_N^{(k-1)} \subseteq K_i \subseteq \Delta_N^{(k)}$$

where $\Delta_N^{(k)}$ is the k-dimensional skeleton of Δ_N.

Suppose that $N \geq (r-1)(d+2)$.

Theorem A conclusion

Then for each continuous map $f : \Delta_N \to \mathbb{R}^d$, there exist vertex-disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that

$$f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$$

and

$$\sigma_1 \in K_1, \sigma_2 \in K_2, \ldots, \sigma_r \in K_r\,.$$

[JPZ-1] D. Jojić, G. Panina, R. Živaljević, *A Tverberg type theorem for collectively unavoidable complexes*, *Israel J. Math.* 2021

Collectively unavoidable complexes and moment-angle complexes

Collectively unavoidable families $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ admit a characterization in the language of generalized moment-angle complexes.

Proposition: Let X be a topological space and $\{A_i\}_{i=1}^r$ a family of its subspaces which are *complementary* in the sense that $X = A_i \cup A_j$ for each $i \neq j$. Then if $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \dots, K_r \rangle$ is a collectively r-unavoidable family of subcomplexes of the N-dimensional simplex $\Delta_N = 2^{[N+1]}$ then

$$X^{N+1} = \mathcal{Z}_{K_1}(X, A_1) \cup \cdots \cup \mathcal{Z}_{K_r}(X, A_r) \,. \tag{20}$$

Conversely, if (20) holds for each X and each family $\{A_i\}_{i=1}^r$ of complementary subspaces in X then $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ is a collectively r-unavoidable family of simplicial complexes.

Proof of the Proposition

It follows from the definition that

$$\mathcal{Z}_{K_i}(X, A_i) = \{x \in X^{N+1} \mid M_i(x) \in K_i\}$$

where $M_i(x) := \{j \in [N+1] \mid x_j \notin A_i\}$.
$A_i \cup A_j = X$ for each $i \neq j$ implies $M_i(x) \cap M_j(x) = \emptyset$. By collective unavoidability of $\mathcal{K} = \langle K_i \rangle_{i=1}^r$, for each $x \in X^{N+1}$ there exists $i \in [r]$ such that $\{M_i(x) \in K_i\}$, and the relation (20) is an immediate consequence.

Conversely, assume that $\mathcal{K} = \langle K_i \rangle_{i=1}^r$ is not collectively unavoidable. By definition there exist pairwise disjoint subsets $\{M_j\}_{j=1}^r$ of $[N+1]$ such that $M_i \notin K_i$ for each $i \in [r]$. Let $X = [N+1]$ and let $A_i := [N+1] \setminus M_i$. Let $x : [N+1] \to X$ be the identity map, ($x_i = i$ for each $i \in [N+1]$). Then,

$$x \in X^{N+1} \setminus \bigcup_{i=1}^r \mathcal{Z}_{K_i}(X, A_i)\,.$$

A canonical family of complementary sets

Let $W = \bigvee_{j=1}^{m} I_j = \bigvee_{j=1}^{m} [0,1]_j$ be the Kowalski *m-hedgehog space* obtained by gluing m "spikes" along 0. Let W_i are its $(m-1)$-hedgehog subspaces obtained by removing the spike I_i.

Then $\{W_i\}_{i=1}^{m}$ is a family of complementary set and if $\mathcal{K} = \langle K_i \rangle_{i=1}^{r} = \langle K_1, \ldots, K_r \rangle$ is a collectively r-unavoidable family of complexes then

$$W^{N+1} = \mathcal{Z}_{K_1}(W, W_1) \cup \cdots \cup \mathcal{Z}_{K_r}(W, W_r)\,. \qquad (21)$$

A central role is played by high connectivity results as illustrated by:

Theorem: Suppose that $\mathcal{K} = \langle K_i \rangle_{i=1}^r = \langle K_1, \ldots, K_r \rangle$ is a *collectively r-unavoidable* family of subcomplexes of $2^{[m]}$. Then the associated deleted join

$$DelJoin(\mathcal{K}) = K_1 *_\Delta K_2 *_\Delta \cdots *_\Delta K_r$$

is $(m - r - 1)$-connected.

D. Jojić, I. Nekrasov, G. Panina, R. Živaljević, *Alexander r-tuples and Bier complexes*, Publ. Inst. Math. (Beograd) (N.S.) 104(118) (2018), 1–22.

Recall that $DelJoin(\mathcal{K})$ is a **generalized chessboard complex**.

Connection with polyhedral products

Theorem:

$$Bier(K) := K *_{\Delta} K^{\circ} \simeq \check{\mathcal{Z}}_K(X; A) \cap \check{\mathcal{Z}}_{K^{\circ}}(X; B)\,.$$

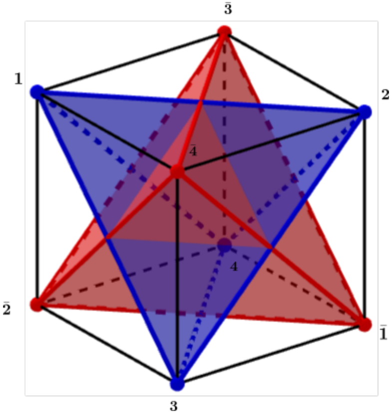
where $X = [0,1], A = [0,1/2], B = [1/2,1]$ and $\check{\mathcal{Z}}_K(X, A) := \mathcal{Z}_K(X, A) \setminus \{1/2\}^m$ is the "reduced" moment-angle complex.

Connection with moment-angle complexes

Theorem:

$$K_1 *_\Delta \cdots *_\Delta K_r \simeq \check{\mathcal{Z}}_{K_1}(W; W_1) \cap \cdots \cap \check{\mathcal{Z}}_{K_r}(W; W_r)$$

where $W = \bigvee_{i=1}^{m}[0,1]$ is the Kowalski *m-hedgehog space* and W_i are its $(m-1)$-hedgehog subspaces. The reduced moment-angle complex is obtained by removing the point $(0, 0, \ldots, 0)$.



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